Introduction
In this unit, students investigate the six trigonometric functions, both as ratios in right triangles and as circular functions, which they graph. The Law of Sines and the Law of Cosines are used to solve problems, as are the inverse trigonometric functions.

The unit concludes with lessons in which students verify and use trigonometric identities, and solve trigonometric equations.

Assessment Options

Unit 5 Test  Pages 899–900 of the Chapter 14 Resource Masters may be used as a test or review for Unit 5. This assessment contains both multiple-choice and short answer items.

TestCheck and Worksheet Builder
This CD-ROM can be used to create additional unit tests and review worksheets.
"The groans from the trigonometry students immediately told teacher Michael Buchanan what the class thought of his idea to read Homer Hickam’s *October Sky*. In the story, in order to accomplish what they would like, the kids had to teach themselves trig, calculus, and physics.” In this project, you will research applications of trigonometry as it applies to a possible career for you.

Log on to [www.algebra2.com/webquest](http://www.algebra2.com/webquest). Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 5.
Trigonometric Functions
Chapter Overview and Pacing

LESSON OBJECTIVES

13–1 Right Triangle Trigonometry (pp. 700–708)
   - Preview: Special Right Triangles
   - Find values of trigonometric functions for acute angles.
   - Solve problems involving right triangles.
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 2 (with 13-1 Preview) 2 (with 13-1 Preview)
     Block: 1 1

13–2 Angles and Angle Measure (pp. 709–716)
   - Change radian measure to degree measure and vice versa.
   - Identify coterminal angles.
   - Follow-Up: Investigating Regular Polygons Using Trigonometry
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 1 1
     Block: 0.5 0.5

13–3 Trigonometric Functions of General Angles (pp. 717–724)
   - Find values of trigonometric functions for general angles.
   - Use reference angles to find values of trigonometric functions.
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 2 (with 13-2 Follow-Up) 2 (with 13-2 Follow-Up)
     Block: 1 1

13–4 Law of Sines (pp. 725–732)
   - Solve problems by using the Law of Sines.
   - Determine whether a triangle has one, two, or no solutions.
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 2
     Block: 1 1

13–5 Law of Cosines (pp. 733–738)
   - Solve problems by using the Law of Cosines.
   - Determine whether a triangle can be solved by first using the Law of Sines or the Law of Cosines.
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 2
     Block: 1 1

13–6 Circular Functions (pp. 739–745)
   - Define and use the trigonometric functions based on the unit circle.
   - Find the exact values of trigonometric functions of angles.
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 2
     Block: 1 1

13–7 Inverse Trigonometric Functions (pp. 746–751)
   - Solve equations by using inverse trigonometric functions.
   - Find values of expressions involving trigonometric functions.
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 1
     Block: 0.5 0.5

Study Guide and Practice Test (pp. 752–757)
Standardized Test Practice (pp. 758–759)
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 1
     Block: 0.5 0.5

Chapter Assessment
   - PACING (days)
     Regular: Basic/ Advanced
     Block: Basic/ Advanced
     Regular: 1
     Block: 1 0.5

TOTAL 14 14 7.5 7

Pacing suggestions for the entire year can be found on pages T20–T21.
### Chapter Resource Manager

#### CHAPTER 13 RESOURCE MASTERS

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*Key to Abbreviations: GCS = Graphing Calculator and Spreadsheet Masters, SC = School-to-Career Masters, SM = Science and Mathematics Lab Manual*
Chapter 13

Mathematical Connections and Background

Continuity of Instruction

Prior Knowledge
Students have used the Pythagorean Theorem. They are familiar with applying formulas to solve problems. Also, they introduced new notation for inverse functions when they explored logarithmic functions, and they restricted domains when they found inverses for functions such as \( y = x^2 \).

This Chapter
Students explore trigonometric functions, first for acute angles in right triangles, then for angles in standard form, and also for points on the unit circle. They derive and use the Law of Sines and the Law of Cosines as applications of trigonometric functions, and they develop inverses for the sine, cosine, and tangent functions.

Future Connections
Students’ exploration of trigonometric functions and periodic functions continues in the following chapter. There they will explore amplitude and frequency for periodic functions and will look at translations of their graphs. They will develop and use trigonometric functions for sums or differences of angles and they will solve equations involving trigonometric terms.

13-1 Right Triangle Trigonometry
To solve many of the problems in this lesson, students will need to remember the Pythagorean Theorem, which states that the sum of the squares of the legs of a right triangle equals the square of the hypotenuse. The hypotenuse is the side directly across from the right angle. It is also the longest side. The legs of a right triangle are the two shorter sides. If the legs of a right triangle have measures \( a \) and \( b \) and the hypotenuse has a measure of \( c \), then \( a^2 + b^2 = c^2 \).

When solving real-world problems involving right triangles and trigonometric ratios, first determine what information is given about the triangle’s sides and angles. Then determine how the given side or sides of the triangle relate to the given angle or the angle to be considered. In other words, determine if a given side is the hypotenuse, side opposite, or side adjacent to the angle. For example, if the problem gives an angle measure and the measure of the side adjacent to this angle and asks you to find the side opposite this angle, use the tangent ratio.

13-2 Angles and Angle Measure
This lesson will introduce students to concept of negative angle measure. It is important to note that an angle measuring \(-210^\circ\) is not less than an angle measuring 210°. The negative angle measure indicates that the direction of the rotation is clockwise instead of counterclockwise.

Remember that in trigonometry an angle is a measure of rotation. One complete rotation, or turn, measures 360°, the number of degrees in a circle. In the real world, objects can rotate about a fixed point many times. Angles measuring more than 360° are used to describe these rotations. To draw such an angle, first subtract 360° from the angle measure and continue doing so until you arrive at a measure that is less than or equal to 360°. Draw this angle and then use an arrow spiraling from the angle’s initial side, through as many 360°-increments as you subtracted, until finally reaching the angle’s terminal side.

13-3 Trigonometric Functions of General Angles
In Lesson 13-1, students find the exact values of the six trigonometric functions for angles measuring less than 90°, since the angles other than the right angle in a right triangle are both acute angles. Right triangle trigonometry is also used to define the values of the trigonometric functions for angles other than acute angles. From a point \( P(x, y) \) on the terminal side of any angle, draw a
segment perpendicular, meeting at a right angle, to the x-axis. A right triangle is formed with one side measuring x units, another side measuring y units, and the hypotenuse measuring r units. The value of r can be found using the Pythagorean Theorem, \( r = \sqrt{x^2 + y^2} \). To determine the values of the six trigonometric functions for the original angle \( \theta \), find the value of each function for the angle \( \theta' \) that is formed by the terminal side and the x-axis. It is important to note that the trigonometric values of angles other than acute angles can be negative.

### 13-4 Law of Sines

Examine the chart on page 727. In each drawing where \( A \) is acute, the positions of the horizontal segment and segment \( b \) are fixed, thus allowing the measure of \( A \) to remain constant. The measure of segment \( b \) in each drawing is also constant. The position of segment \( a \), however, can change like a door on a hinge. Notice also that the length of \( a \) in each diagram is different and often compared to the value \( b \sin A \).

In the first diagram in the first row, the length of \( a \) is less than the value \( b \sin A \). In other words, side \( a \) is too short to form a right triangle. When \( a \) is too short to form a right triangle, no triangle can be formed. When \( a \) equals \( b \sin A \), a right triangle is formed, as in the second diagram. In the first diagram in the second row, \( a \) is greater than \( b \sin A \) but still less than \( b \). In other words, side \( a \) is too long to form a right triangle, but can still rotate on its hinge and meet the opposite side to form either an obtuse or an acute triangle. If \( a \) is greater than or equal to \( b \), side \( a \) is again too long to form a right triangle, but is now also too long to rotate back towards \( \angle A \). Instead, it can meet the opposite side in only one place, as shown in the second diagram in the second row.

### 13-5 Law of Cosines

The Law of Cosines involves three sides and one angle of a triangle. The key step in deriving the Law of Cosines is to consider an altitude of length \( h \) intersecting a side and dividing it into two parts. Using \( x \) to represent one of the two parts, the Pythagorean Theorem gives an equation in terms of \( x \) and the three side lengths. Then \( x \) can be replaced with an expression involving one side and the cosine of one angle. The result is an expression of the Law of Cosines as three equations:

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos A \\
    b^2 &= a^2 + c^2 - 2ac \cos B \\
    c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

The Law of Cosines lets you calculate measures for all the angles and all the sides of a triangle if you are given (1) the length of two sides and the measure of the included angle, or (2) the lengths of three sides. The Law of Sines can be used if the measures of two angles and one side are known (ASA, AAS) or if the measures of two sides and the non-included angle are known (SSA, but there may be 0, 1, or 2 triangles).

### 13-6 Circular Functions

To memorize the unit circle presented at the bottom of page 740, use the following technique. First memorize the coordinates for the angles in the first quadrant. Notice that all of these coordinates contain values having a denominator of 2, that the \( x \) - and \( y \)-coordinates for 45° are identical, \( \frac{\sqrt{2}}{2} \), and that the \( x \) - and \( y \)-coordinates for 30° and 60° are reversed. With this quadrant memorized, the coordinates for the other quadrants can be obtained using the signs of \( x \) and \( y \) in each quadrant and the symmetry of a circle.

The coordinates of the quadrantal angles, 0°, 90°, 180°, and 270°, are easily remembered by recalling that a unit circle has a radius of 1 unit. For example, since 0° is located on the x-axis, its coordinates are (1, 0).

Once the unit circle is completed, it can be used as a reference to obtain the sine or cosine of any of the angles listed. One need only remember that the \( x \)-coordinate given for an angle measure is the cosine of the angle and the \( y \)-coordinate is the sine of the angle. In other words \( (x, y) = (\cos \theta, \sin \theta) \).

### 13-7 Inverse Trigonometric Functions

In this lesson, students limit the domain of the sine, cosine, and tangent functions, and define an inverse for each function. The graphs of those functions do not pass the horizontal line test, so they would not have inverses. However, the section of the sine and tangent graphs between -180° and 180° and the section of the cosine graph between 0° and 180° do pass the horizontal line test, so inverse functions can be identified for those domains. The function \( y = \sin x \) is the restricted-domain sine function. Its inverse is \( x = \sin^{-1} y \) or \( x = \arcsin y \). The function \( y = \cos x \) is the restricted-domain cosine function. Its inverse is \( x = \cos^{-1} y \) or \( x = \arccos y \). The function \( y = \tan x \) represents the restricted-domain tangent function. Its inverse function is \( x = \tan^{-1} y \) or \( x = \arctan y \).
### Chapter 13: Trigonometric Functions

**Additional Intervention Resources**
The Princeton Review’s *Cracking the SAT & PSAT*
The Princeton Review’s *Cracking the ACT*
ALEKS

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|                    | Quizzes, *CRM* pp. 831–832  
|                    | Mid-Chapter Test, *CRM* p. 833  
|                    |                                                        | www.algebra2.com/self_check_quiz  
|                    |                                                        | www.algebra2.com/extra_examples |
| Error Analysis     | Find the Error, pp. 730, 735  
|                    | Common Misconceptions, p. 703                          | Find the Error, *TWE* pp. 730, 735  
|                    |                                                        | Unlocking Misconceptions, *TWE* pp. 718, 726  
|                    |                                                        | Tips for New Teachers, *TWE* pp. 703, 711                                           |
| Standardized Test Practice | pp. 702, 706, 708, 714, 724, 732, 737, 738, 745, 751, 757, 758–759 | *TWE* p. 702  
|                    |                                                        | Standardized Test Practice, *CRM* pp. 835–836                                      | Standardized Test Practice  
|                    |                                                        | CD-ROM                                                                            | www.algebra2.com/standardized_test |
| Open-Ended Assessment | Writing in Math, pp. 708, 714, 724, 732, 737, 744, 751  
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|                    |                                                        | Writing: *TWE* pp. 724, 738  
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| Chapter Assessment | Study Guide, pp. 752–756  
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|                    |                                                        | Free-Response Tests (Forms 2C, 2D, 3), *CRM* pp. 823–828  
|                    |                                                        | Vocabulary Test/Review, *CRM* p. 830                                                | TestCheck and Worksheet Builder  
|                    |                                                        | (see below)                                                                        | www.algebra2.com/chapter_test |

*Key to Abbreviations:* TWE = Teacher Wraparound Edition; CRM = Chapter Resource Masters

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**TestCheck and Worksheet Builder**

This networkable software has three modules for intervention and assessment flexibility:
- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.
**Intervention Technology**

*Alge2PASS: Tutorial Plus* CD-ROM offers a complete, self-paced algebra curriculum.

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**ALEKS** is an online mathematics learning system that adapts assessment and tutoring to the student’s needs. Subscribe at [www.k12aleks.com](http://www.k12aleks.com).

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**Intervention at Home**

*Log on for student study help.*

- For each lesson in the Student Edition, there are Extra Examples and Self-Check Quizzes.  
  [www.algebra2.com/extra_examples](http://www.algebra2.com/extra_examples)  
  [www.algebra2.com/self_check_quiz](http://www.algebra2.com/self_check_quiz)
- For chapter review, there is vocabulary review, test practice, and standardized test practice.  
  [www.algebra2.com/vocabulary_review](http://www.algebra2.com/vocabulary_review)  
  [www.algebra2.com/chapter_test](http://www.algebra2.com/chapter_test)  
  [www.algebra2.com/standardized_test](http://www.algebra2.com/standardized_test)

*For more information on Intervention and Assessment, see pp. T8–T11.*

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**Reading and Writing in Mathematics**

*Glencoe Algebra 2* provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

**Student Edition**

- Foldables Study Organizer, p. 699
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 706, 712, 722, 729, 735, 742, 749, 752)
- Writing in Math questions in every lesson, pp. 708, 714, 724, 732, 737, 744, 751
- Reading Study Tip, pp. 701, 709, 711, 718, 740
- WebQuest, p. 708

**Teacher Wraparound Edition**

- Foldables Study Organizer, p. 699, 752
- Study Notebook suggestions, pp. 706, 713, 716, 722, 730, 735, 743, 749
- Modeling activities, pp. 708, 732, 745
- Speaking activities, pp. 715, 751
- Writing activities, pp. 724, 738
- Differentiated Instruction, (Verbal/Linguistic), p. 735
- **ELL** Resources, pp. 698, 707, 714, 723, 731, 735, 737, 744, 750, 752

**Additional Resources**

- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (*Chapter 13 Resource Masters*, pp. vii–viii)
- Reading to Learn Mathematics master for each lesson (*Chapter 13 Resource Masters*, pp. 779, 785, 791, 797, 803, 809, 815)
- **Vocabulary PuzzleMaker** software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- *Teaching Mathematics with Foldables* provides suggestions for promoting cognition and language.
- Reading and Writing in the Mathematics Classroom
- WebQuest and Project Resources

*For more information on Reading and Writing in Mathematics, see pp. T6–T7.*
Have students read over the list of objectives and make a list of any words with which they are not familiar.

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

Trigonometry is the study of the relationships among the angles and sides of right triangles. One of the many real-world applications of trigonometric functions involves solving problems using indirect measurement. For example, surveyors use a trigonometric function to find the heights of buildings. You will learn how architects who design fountains use a trigonometric function to aim the water jets in Lesson 13-7.

The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the Chapter 13 Resource Masters. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 13 test.
This section provides a review of the basic concepts needed before beginning Chapter 13. Page references are included for additional student help.

**Prerequisite Skills** in the Getting Ready for the Next Lesson section at the end of each exercise set review a skill needed in the next lesson.

### Prerequisite Skills

For Lessons 13-1 and 13-3

#### Pythagorean Theorem

Find the value of x to the nearest tenth. (For review, see pages 820 and 821.)

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<td>3.</td>
<td>16.7</td>
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<tr>
<td>4.</td>
<td>21.8</td>
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#### 45°-45°-90° and 30°-60°-90° Triangles

Find each missing measure. Write all radicals in simplest form.

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<tr>
<td>5.</td>
<td>( x = 7, y = 7\sqrt{2} )</td>
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<tr>
<td>6.</td>
<td>( x = \frac{21\sqrt{2}}{2} )</td>
</tr>
<tr>
<td>7.</td>
<td>( x = 4\sqrt{3}, y = 8 )</td>
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<tr>
<td>8.</td>
<td>( x = 3\sqrt{3}, y = 6\sqrt{3} )</td>
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For Lesson 13-7

#### Inverse Functions

Find the inverse of each function. Then graph the function and its inverse. (For review, see Lesson 7-8.) 9–12. See pp. 759A–759D for graphs.

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<td>9.</td>
<td>( f(x) = x + 3 ) ( f^{-1}(x) = x - 3 )</td>
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<tr>
<td>10.</td>
<td>( f(x) = x - \frac{2}{5} ) ( f^{-1}(x) = 5x + 2 )</td>
</tr>
<tr>
<td>11.</td>
<td>( f(x) = x^2 - 4 ) ( f^{-1}(x) = \pm \sqrt{x + 4} )</td>
</tr>
<tr>
<td>12.</td>
<td>( f(x) = -7x - 9 ) ( f^{-1}(x) = \frac{-x - 9}{7} )</td>
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### Foldables Study Organizer

Make this Foldable to help you organize information about trigonometric functions. Begin with one sheet of construction paper and two pieces of grid paper.

**Step 1** Fold and Cut

Stack and fold on the diagonal. Cut to form a triangular stack.

**Step 2** Staple and Label

Staple edge to form a booklet.

### Reading and Writing

As you read and study the chapter, you can write notes, draw diagrams, and record formulas on the grid paper.

### Organization of Data: Vocabulary and Visuals

Have students use their right triangle journals to practice writing concise definitions in their own words and to design visuals that present the information introduced in the lesson in a concrete, easy-to-study format. Encourage students to clearly label their visuals and write captions when needed. Students can use this study guide to review what they know and apply it to what they are currently learning.
Spreadsheet Investigation

A Preview of Lesson 13-1

Getting Started

Cell Format  Students can format the cells before they start to enter the data by using the Format Cells command. Format columns C, E, and F for numbers with 8 decimal places. Format columns A, B, and D for integers.

Teach

- Make sure students understand how to enter the formulas shown for columns C through F.
- Have students practice their skills in using a spreadsheet by entering the data in the example.
- Have students complete Exercises 1–3.
- To extend this investigation, ask students to explore the effect on the ratios if \( \frac{a}{H11005} \) when \( a \) and \( b \) are rational numbers (such as 3.6) instead of integers.

Assess

Ask students:

- What formula could you have used in column B for the 45°-45°-90° triangle instead of entering the data? in column C for the 30°-60°-90° triangle? \( b = a; c = 2a \)
- Compare the 30°-60°-90° triangles. What is the same? What is different? The angle measures are all the same, but the side measures are different. The triangles are similar but not congruent.

Answer

3. All of the ratios of side \( b \) to side \( a \) are approximately 1.73. All of the ratios of side \( b \) to side \( c \) are approximately 0.87. All of the ratios of side \( a \) to side \( c \) are 0.5.
**Study Tip**

The word trigonometry is derived from two Greek words—trigon meaning triangle and metra meaning measurement.

**Vocabulary**

- trigonometry
- trigonometric functions
- sine
- cosine
- tangent
- secant
- cosecant
- cotangent
- solve a right triangle
- angle of elevation
- angle of depression

**What You’ll Learn**

- Find values of trigonometric functions for acute angles.
- Solve problems involving right triangles.

**How is trigonometry used in building construction?**

The Americans with Disabilities Act (ADA) provides regulations designed to make public buildings accessible to all. Under this act, the slope of an entrance ramp designed for those with mobility disabilities must not exceed a ratio of 1 to 12. This means that for every 12 units of horizontal run, the ramp can rise or fall no more than 1 unit.

When viewed from the side, a ramp forms a right triangle. The slope of the ramp can be described by the tangent of the angle the ramp makes with the ground. In this example, the tangent of angle A is \( \frac{1}{12} \).

**TRIGONOMETRIC VALUES**

The tangent of an angle is one of the ratios used in trigonometry. Trigonometry is the study of the relationships among the angles and sides of a right triangle.

Consider right triangle \( ABC \) in which the measure of acute angle \( A \) is identified by the Greek letter \( \theta \). The sides of the triangle are the hypotenuse, the leg opposite \( \theta \), and the leg adjacent to \( \theta \).

Using these sides, you can define six trigonometric functions: sine, cosine, tangent, secant, cosecant, and cotangent. These functions are abbreviated \( \sin, \cos, \tan, \sec, \csc, \) and \( \cot \), respectively.

**Key Concept**

If \( \theta \) is the measure of an acute angle of a right triangle, \( \text{opp} \) is the measure of the leg opposite \( \theta \), \( \text{adj} \) is the measure of the leg adjacent to \( \theta \), and \( \text{hyp} \) is the measure of the hypotenuse, then the following are true.

\[
\begin{align*}
\sin \theta &= \frac{\text{opp}}{\text{hyp}} & \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \tan \theta &= \frac{\text{opp}}{\text{adj}} \\
\csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta}
\end{align*}
\]

Notice that the sine, cosine, and tangent functions are reciprocals of the cosecant, secant, and cotangent functions, respectively. Thus, the following are also true.

**5-Minute Check Transparency 13-1**

Use as a quiz or review of Chapter 12.

**Mathematical Background**

In previous courses, students learned about the Pythagorean Theorem. In this lesson, students will apply their knowledge to solve triangles.

**Ask students:**

- To meet the required ratio, how long would a ramp have to be to rise from the ground to a door that is 2 feet above ground level? \( 24 \text{ ft} \)
- Does a ramp with a ratio of 0.08 meet the requirement? \( \text{yes} \)

**Workbook and Reproducible Masters**

- Graphing Calculator and Spreadsheet Masters, p. 52
- Science and Mathematics Lab Manual, pp. 37–40
- Teaching Algebra With Manipulatives Masters, pp. 299–300
The domain of each of these trigonometric functions is the set of all acute angles \( \theta \) of a right triangle. The values of the functions depend only on the measure of \( \theta \) and not on the size of the right triangle. For example, consider \( \sin \theta \) in the figure at the right.

Using \( \triangle ABC \):

\[
\sin \theta = \frac{BC}{AB}.
\]

Using \( \triangle AB'C' \):

\[
\sin \theta = \frac{B'C'}{AB'}.
\]

The right triangles are similar because they share angle \( \theta \). Since they are similar, the ratios of corresponding sides are equal. That is, \( \frac{BC}{AB} = \frac{B'C'}{AB'} \). Therefore, you will find the same value for \( \sin \theta \) regardless of which triangle you use.

### Example 1 Find Trigonometric Values

Find the values of the six trigonometric functions for angle \( \theta \).

For this triangle, the leg opposite \( \theta \) is \( AB \), and the leg adjacent to \( \theta \) is \( CB \). Recall that the hypotenuse is always the longest side of a right triangle, in this case \( AC \).

Use \( \text{opp} = 4 \), \( \text{adj} = 3 \), and \( \text{hyp} = 5 \) to write each trigonometric ratio.

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}, \quad \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}.
\]

\[
\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}, \quad \sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}, \quad \cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}.
\]

Throughout Unit 5, a capital letter will be used to denote both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase will be used to denote the side opposite that angle and its measure.

### Example 2 Use One Trigonometric Ratio to Find Another

#### Multiple-Choice Test Item

If \( \cos A = \frac{2}{5} \), find the value of \( \tan A \).

- (A) \( \frac{5}{2} \)
- (B) \( 2\sqrt{21} \)
- (C) \( \frac{\sqrt{21}}{2} \)
- (D) \( \sqrt{21} \)

#### Read the Test Item

Begin by drawing a right triangle and labeling one acute angle \( A \). Since \( \cos \theta = \frac{\text{adj}}{\text{hyp}} \) and \( \cos A = \frac{2}{5} \) in this case, label the adjacent leg 2 and the hypotenuse 5.

#### Solve the Test Item

Use the Pythagorean Theorem to find \( a \).

\[
a^2 + b^2 = c^2 \quad \text{Pythagorean Theorem}
\]

\[
a^2 + 2^2 = 5^2 \quad \text{Replace } b \text{ with 2 and } c \text{ with 5}.
\]

\[
a^2 + 4 = 25 \quad \text{Simplify}.
\]

\[
a^2 = 21 \quad \text{Subtract 4 from each side}.
\]

\[
a = \sqrt{21} \quad \text{Take the square root of each side}.
\]

Then write \( \tan A = \frac{\text{opp}}{\text{adj}} \). This suggests looking for an answer choice that has a denominator of 2.
Angles that measure 30°, 45°, and 60° occur frequently in trigonometry. The table below gives the values of the six trigonometric functions for these angles. To remember these values, use the properties of 30°-60°-90° and 45°-45°-90° triangles.

### Trigonometric Values for Special Angles

<table>
<thead>
<tr>
<th>θ</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
<th>csc θ</th>
<th>sec θ</th>
<th>cot θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>1/2</td>
<td>√3/2</td>
<td>√3</td>
<td>2</td>
<td>2√3/3</td>
<td>√3</td>
</tr>
<tr>
<td>45°</td>
<td>√2/2</td>
<td>√2/2</td>
<td>1</td>
<td>√2</td>
<td>√2</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>√3/2</td>
<td>1/2</td>
<td>√3</td>
<td>2√3/3</td>
<td>2</td>
<td>√3/3</td>
</tr>
</tbody>
</table>

You will verify some of these values in Exercises 27 and 28.

### RIGHT TRIANGLE PROBLEMS

You can use trigonometric functions to solve problems involving right triangles.

#### Example 3 Find a Missing Side Length of a Right Triangle

Write an equation involving sin, cos, or tan that can be used to find the value of x. Then solve the equation. Round to the nearest tenth.

The measure of the hypotenuse is 8. The side with the missing length is adjacent to the angle measuring 30°. The trigonometric function relating the adjacent side of a right triangle and the hypotenuse is the cosine function.

\[
\cos \theta = \frac{\text{adj}}{\text{hyp}}
\]

\[
\cos 30° = \frac{x}{8}
\]

Replace \( \theta \) with 30°, adj with \( x \), and hyp with 8.

\[
\frac{\sqrt{3}}{2} = \frac{x}{8}
\]

\[
x = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}
\]

The value of \( x \) is 4\( \sqrt{3} \) or about 6.9.

A calculator can be used to find the value of trigonometric functions for any angle, not just the special angles mentioned. Use [SIN], [COS], and [TAN] for sine, cosine, and tangent. Use these keys and the reciprocal key, [1/x], for cosecant, secant, and cotangent. Be sure your calculator is in degree mode.

Now find tan A.

\[
\tan A = \frac{\text{opp}}{\text{adj}}
\]

\[
= \frac{\sqrt{21}}{2}
\]

Replace opp with \( \sqrt{21} \) and adj with 2.

The answer is C.
Solve $\triangle XYZ$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

$Y = 62^\circ$, $x \approx 5.8$, $z \approx 12.5$

Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

$A \approx 32^\circ$, $B = 58^\circ$

What is the minimum information you have to have about a right triangle to solve it? One of the following: measure of an acute angle, length of a leg, length of the hypotenuse.

Here are some calculator examples.

$\cos 46^\circ$ \hspace{10pt}\text{KEYSTROKES:} \hspace{5pt} \cos 46 \hspace{5pt} \text{ENTER} \hspace{5pt} 0.6946583705$

$\cot 20^\circ$ \hspace{10pt}\text{KEYSTROKES:} \hspace{5pt} \tan 20 \hspace{5pt} \text{ENTER} \hspace{5pt} x^- \hspace{5pt} \text{ENTER} \hspace{5pt} 2.74774719$

If you know the measures of any two sides of a right triangle or the measures of one side and one acute angle, you can determine the measures of all the sides and angles of the triangle. This process of finding the missing measures is known as solving a right triangle.

**Example 4** Solve a Right Triangle

Solve $\triangle XYZ$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

You know the measures of one side, one acute angle, and the right angle. You need to find $x$, $z$, and $Y$.

Find $x$ and $z$.

$\tan 35^\circ = \frac{x}{10}$

$10 \tan 35^\circ = x$

$7.0 = x$

$\sec 35^\circ = \frac{z}{10}$

$\frac{1}{\cos 35^\circ} = \frac{z}{10}$

$12.2 = z$

Find $Y$.

$35^\circ + Y = 90^\circ$ Angles $X$ and $Y$ are complementary.

$Y = 55^\circ$ Solve for $Y$.

Therefore, $Y = 55^\circ$, $x = 7.0$, and $z = 12.2$.

Use the inverse capabilities of your calculator to find the measure of an angle when one of its trigonometric ratios is known. For example, use the $\sin^{-1}$ function to find the measure of an angle when the sine of the angle is known. You will learn more about inverses of trigonometric functions in Lesson 13-7.

**Example 5** Find Missing Angle Measures of Right Triangles

Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

You know the measures of the sides. You need to find $A$ and $B$.

Find $A$.

$\sin A = \frac{5}{13}$ \hspace{10pt}\text{Sin A = \frac{opp}{hyp}}$

Use a calculator and the $\sin^{-1}$ function to find the angle whose sine is $\frac{5}{13}$.

\text{KEYSTROKES:} \hspace{5pt} 2nd \hspace{5pt} [\sin^{-1}] \hspace{5pt} 5 \hspace{5pt} \pm \hspace{5pt} 13 \hspace{5pt} \text{ENTER} \hspace{5pt} 22.61986495$

To the nearest degree, $A = 23^\circ$.

Find $B$.

$23^\circ + B = 90^\circ$ Angles $A$ and $B$ are complementary.

$B = 67^\circ$ Solve for $B$.

Therefore, $A = 23^\circ$ and $B = 67^\circ$.

**Differentiated Instruction**

**Visual/Spatial** Have students use a stack of books and a notebook to model a ramp and investigate how steep the ramp needs to be for a toy car to roll down it without being pushed. Have them report their results in terms of the trigonometric functions of a right triangle.
Trigonometry has many practical applications. Among the most important is the ability to find distances or lengths that either cannot be measured directly or are not easily measured directly.

### Example 6  Indirect Measurement

**Bridging Construction** In order to construct a bridge across a river, the width of the river at that location must be determined. Suppose a stake is planted on one side of the river directly across from a second stake on the opposite side. At a distance 50 meters to the left of the stake, an angle of 82° is measured between the two stakes. Find the width of the river.

Let \( w \) represent the width of the river at that location. Write an equation using a trigonometric function that involves the ratio of the distance \( w \) and 50.

\[
\tan 82^\circ = \frac{w}{50}
\]

Multiply each side by 50.

\[
355.8 = w
\]

Use a calculator.

The width of the river is about 355.8 meters.

### Example 7  Use an Angle of Elevation

**Skiing** The Aerial run in Snowbird, Utah, has an angle of elevation of 20.2°. Its vertical drop is 2900 feet. Estimate the length of this run.

Let \( \ell \) represent the length of the run. Write an equation using a trigonometric function that involves the ratio of \( \ell \) and 2900.

\[
\sin 20.2^\circ = \frac{2900}{\ell}
\]

Solve for \( \ell \).

\[
\ell = \frac{2900}{\sin 20.2^\circ}
\]

Use a calculator.

The length of the run is about 8399 feet.
1. Trigonometry is the study of the relationships between the angles and sides of a right triangle.

2. 

3. Given only the measures of the angles of a right triangle, you cannot find the measures of its sides.

4. \[ \sin \theta = \frac{8}{17}; \cos \theta = \frac{15}{17}; \tan \theta = \frac{8}{15}; \csc \theta = \frac{17}{8}; \sec \theta = \frac{17}{15}; \cot \theta = \frac{15}{8} \]

5. \[ \sin \theta = \frac{\sqrt{85}}{11}; \cos \theta = \frac{6}{11}; \tan \theta = \frac{\sqrt{85}}{6}; \csc \theta = \frac{11\sqrt{85}}{85}; \sec \theta = \frac{11}{6}; \cot \theta = \frac{6\sqrt{85}}{85} \]

6. \[ \sin \theta = \frac{5}{6}; \cos \theta = \frac{\sqrt{11}}{6}; \tan \theta = \frac{5\sqrt{11}}{11}; \csc \theta = \frac{6}{5}; \sec \theta = \frac{6\sqrt{11}}{11}; \cot \theta = \frac{\sqrt{11}}{5} \]
21. \( \tan 30^\circ = \frac{x}{10}; \quad x = 5.8 \)
22. \( \cos 60^\circ = \frac{3}{x}; \quad x = 6 \)
23. \( \sin 54^\circ = 17.8; \quad x = 22.0 \)
24. \( \tan 17.5^\circ = \frac{x}{23.7}; \quad x = 7.5 \)
25. \( \cos x^\circ = 15; \quad x = 36^\circ \)
26. \( \sin x^\circ = 16; \quad x = 47^\circ \)

27. Using the 30°-60°-90° triangle shown on page 703, verify each value.
   a. \( \sin 30^\circ = \frac{1}{2} \)
   b. \( \cos 30^\circ = \frac{\sqrt{3}}{2} \)
   c. \( \sin 60^\circ = \frac{\sqrt{3}}{2} \)

28. Using the 45°-45°-90° triangle shown on page 703, verify each value.
   a. \( \sin 45^\circ = \frac{\sqrt{2}}{2} \)
   b. \( \cos 45^\circ = \frac{\sqrt{2}}{2} \)
   c. \( \tan 45^\circ = 1 \)

Solve \( \Delta ABC \) by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. 29–40. See pp. 759A–759D.

29. \( A = 16^\circ, \quad c = 14 \)
30. \( B = 27^\circ, \quad b = 7 \)
31. \( A = 34^\circ, \quad a = 10 \)
32. \( B = 15^\circ, \quad c = 25 \)
33. \( B = 30^\circ, \quad b = 11 \)
34. \( A = 45^\circ, \quad c = 7\sqrt{2} \)
35. \( B = 18^\circ, \quad a = \sqrt{15} \)
36. \( A = 10^\circ, \quad b = 15 \)
37. \( b = 6, \quad c = 13 \)
38. \( a = 4, \quad c = 9 \)
39. \( \tan B = \frac{7}{8}, \quad b = 7 \)
40. \( \sin A = \frac{1}{3}, \quad a = 5 \)

41. TRAVEL In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls? about 300 ft

42. SURVEYING A surveyor stands 100 feet from a building and sights the top of the building at a 55° angle of elevation. Find the height of the building. about 142.8 ft

EXERCISE For Exercises 43 and 44, use the following information.
A preprogrammed workout on a treadmill consists of intervals walking at various rates and angles of incline. A 1% incline means 1 unit of vertical rise for every 100 units of horizontal run.
43. At what angle, with respect to the horizontal, is the treadmill bed when set at a 10% incline? Round to the nearest degree. about 6°
44. If the treadmill bed is 40 inches long, what is the vertical rise when set at an 8% incline? about 3.2 in.

45. GEOMETRY Find the area of the regular hexagon with point \( O \) as its center. (Hint: First find the value of \( x \)) \( 93.54 \text{ units}^2 \)

www.algebra2.com/self_check_quiz

Lesson 13-1 Right Triangle Trigonometry 707

Enrichment, p. 780

The Angle of Repose
Suppose you place a block of wood on an inclined plane, as shown in the figure. If the angle \( \theta \) is small, the plane is inclined from the horizontal by a small angle. As you increase the angle \( \theta \), the block begins to slide down the plane.

The angle of repose is the angle between the plane and the horizontal that the block makes when it starts to slide down the plane.

For situations in which the block and plane are smooth, the angle of repose depends only on the force of gravity acting on the block and the plane. The angle is independent of the area of contact between the block and the plane.

A. Find the area of the triangle formed by the plane and the horizontal. The area is the product of the base of the triangle and the height.
B. Find the angle of repose. The angle is the angle between the plane and the horizontal for which the block just starts to slide down the plane.

Sample answer: The shortest leg is \( \theta \) at an angle as the hypotenuse. You can use the Pythagorean Theorem to find the length of the longer leg.

Helping You Remember
5. In studying trigonometry, it is important for you to remember the relationships between the trigonometric functions of a 45°-45°-90° triangle. If you remember just one fact about the 45°-45°-90° triangle, you will always be able to figure out the lengths of all the sides. What fact can you use? \( \theta \) is a right angle. Use the Pythagorean Theorem to find the length of the longer leg.
Open-Ended Assessment

Modeling Have students design a ramp, specifying the angle and how far the end of the ramp is from the place on the ground where it begins. Have them draw a picture and show steps to find the length of the ramp. Does the ramp meet ADA requirements?

Getting Ready for Lesson 13–2

PREREQUISITE SKILL Lesson 13–2 presents changing between measuring angles in radians and in degrees. Students will use their familiarity with converting units as they write the measures of angles in both radians and degrees. Exercises 59–62 should be used to determine your students’ familiarity with dimensional analysis.

Answers

47. The sine and cosine ratios of acute angles of right triangles each have the longest measure of the triangle, the hypotenuse, as their denominator. A fraction whose denominator is greater than its numerator is less than 1. The tangent ratio of an acute angle of a right triangle does not involve the measure of the hypotenuse, \( \frac{\text{opp}}{\text{adj}} \). If the measure of the opposite side is greater than the measure of the adjacent side, the tangent ratio is greater than 1. If the measure of the opposite side is less than the measure of the adjacent side, the tangent ratio is less than 1.

48. When construction involves right triangles, including building ramps, designing buildings, or surveying land before building, trigonometry is likely to be used. Answers should include the following.
- If you view the ramp from the side then the vertical rise is opposite the angle that the ramp makes with the horizontal. Similarly, the horizontal run is the adjacent side. So the tangent of the angle is the ratio of the rise to the run or the slope of the ramp.
- Given the ratio of the slope of ramp, you can find the angle of inclination by calculating \( \tan^{-1} \) of this ratio.

51. Band members may be more likely to like the same kinds of music.

52. This sample is random since different kinds of people go to the post office.
**5-Minute Check Transparency 13-2** Use as a quiz or review of Lesson 13-1.

**Mathematical Background** notes are available for this lesson on p. 698C.

**How can angles be used to describe circular motion?**

Ask students:

- What is the space between one gondola and the next around the circumference? 11 ft
- How many degrees of the circle are between one gondola and the next? \( \frac{90}{11} \)°
- What is the radius of the Ferris wheel? 70 ft

**ANGLE MEASUREMENT** What does an angle measuring 225° look like?

In Lesson 13-1, you worked only with acute angles, those measuring between 0° and 90°, but angles can have any real number measurement.

On a coordinate plane, an angle may be generated by the rotation of two rays that share a fixed endpoint at the origin. One ray, called the initial side of the angle, is fixed along the positive x-axis. The other ray, called the terminal side of the angle, can rotate about the center. An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis is said to be in standard position.

The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side.

**Positive Angle Measure**

- Counterclockwise

**Negative Angle Measure**

- Clockwise

When terminal sides rotate, they may sometimes make one or more revolutions. An angle whose terminal side has made exactly one revolution has a measure of 360°.
In-Class Example

**Teaching Tip** Discuss with students the importance of marking the drawings of angles with the arrows as shown, and labeling the degrees.

1. Draw an angle with the given measure in standard position.
   a. $210^\circ$
   
   ![Diagram of 210° angle]

   b. $-45^\circ$
   
   ![Diagram of -45° angle]

   c. $540^\circ$
   
   ![Diagram of 540° angle]

**Example 1** Draw an angle with the given measure in standard position.

a. $240^\circ$  
$240^\circ = 180^\circ + 60^\circ$
Draw the terminal side of the angle $60^\circ$ counterclockwise past the negative $x$-axis.

![Diagram showing 240° angle]

b. $-30^\circ$  
The angle is negative.
Draw the terminal side of the angle $30^\circ$ clockwise from the positive $x$-axis.

![Diagram showing -30° angle]

c. $450^\circ$  
$450^\circ = 360^\circ + 90^\circ$
Draw the terminal side of the angle $90^\circ$ counterclockwise past the positive $x$-axis.

![Diagram showing 450° angle]

Another unit used to measure angles is a radian. The definition of a radian is based on the concept of a *unit circle*, which is a circle of radius 1 unit whose center is at the origin of a coordinate system. One radian is the measure of an angle $\theta$ in standard position whose rays intercept an arc of length 1 unit on the unit circle.

The circumference of any circle is $2\pi r$, where $r$ is the radius measure. So the circumference of a unit circle is $2\pi(1)$ or $2\pi$ units. Therefore, an angle representing one complete revolution of the circle measures $2\pi$ radians. This same angle measures $360^\circ$. Therefore, the following equation is true.

$$2\pi \text{ radians} = 360^\circ$$

To change angle measures from radians to degrees or vice versa, solve the equation above in terms of both units.

$$2\pi \text{ radians} = 360^\circ$$

$$\frac{2\pi \text{ radians}}{2\pi} = \frac{360^\circ}{2\pi}$$

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

1 radian is about 57 degrees. 1 degree is about 0.0175 radian.

These equations suggest a method for converting between radian and degree measure.
Example 2  Convert Between Degree and Radian Measure

Rewrite the degree measure in radians and the radian measure in degrees.

a. \(60^\circ\)
\[
60^\circ = \frac{60^\circ \cdot \pi \text{ radians}}{180^\circ} = \frac{60\pi}{180} \text{ radians or } \frac{\pi}{3}
\]

b. \(-\frac{7\pi}{4}\)
\[
-\frac{7\pi}{4} = -\frac{7\pi}{4} \cdot \frac{180^\circ}{\pi} = -\frac{1260^\circ}{4} \text{ or } -315^\circ
\]

You will find it useful to learn equivalent degree and radian measures for the special angles shown in the diagram at the right. This diagram is more easily learned by memorizing the equivalent degree and radian measures for the first quadrant and for \(90^\circ\). All of the other special angles are multiples of these angles.

Example 3  Measure an Angle in Degrees and Radians

TIME  Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 1:00 P.M. to 3:00 P.M.

The numbers on a clock divide it into 12 equal parts with 12 equal angles. The angle from 1 to 3 on the clock represents \(\frac{2}{12}\) or \(\frac{1}{6}\) of a complete rotation of 360°. \(\frac{1}{6}\) of 360° is 60°.

Since the rotation is clockwise, the angle through which the hour hand rotates is negative. Therefore, the angle measures \(-60^\circ\).

\(60^\circ\) has an equivalent radian measure of \(\frac{\pi}{3}\). So the equivalent radian measure of \(-60^\circ\) is \(-\frac{\pi}{3}\).

COTERMINAL ANGLES  If you graph a 405° angle and a 45° angle in standard position on the same coordinate plane, you will notice that the terminal side of the 405° angle is the same as the terminal side of the 45° angle. When two angles in standard position have the same terminal sides, they are called coterminal angles.
COTERMINAL ANGLES

In-Class Example

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

- 210° Sample answers: 570°, −150°
- \( \frac{7\pi}{3} \) Sample answers: \(-\frac{\pi}{3}, -\frac{5\pi}{3}\)

Answers

2. In a circle of radius \( r \) units, one radian is the measure of an angle whose rays intercept an arc length of \( r \) units.

3.

\[ \text{Draw an angle with the given measure in standard position.} \]

4. 290°

5.

6.

7.

8.

9.

Study Tip

Coterminal Angles

Notice in Example 4b that it is necessary to subtract a multiple of \( 2\pi \) to find a coterminal angle with negative measure.

Example 4 Find Coterminal Angles

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

- a. 240°
  A positive angle is 
  \( 240° + 360° \) or 600°.
  A negative angle is 
  \( 240° - 360° \) or −120°.

- b. \( \frac{9\pi}{4} \)
  A positive angle is 
  \( \frac{9\pi}{4} + 2\pi \) or \( \frac{17\pi}{4} \).
  A negative angle is 
  \( \frac{9\pi}{4} - 2(2\pi) \) or \( -\frac{7\pi}{4} \).

Check for Understanding

Concept Check

1. Name the set of numbers to which angle measures belong. reals
2. Define the term radian. See margin.
3. OPEN ENDED Draw and label an example of an angle with negative measure in standard position. Then find an angle with positive measure that is coterminal with this angle. See margin.

Guided Practice

Draw an angle with the given measure in standard position. 4–7. See margin.

Rewrite each degree measure in radians and each radian measure in degrees.

- 8. 130° \( \frac{13\pi}{18} \)
- 9. \(-10°\) \( -\frac{\pi}{18} \)
- 10. 485° \( \frac{97\pi}{36} \)
- 11. \( \frac{3\pi}{4} \)
- 12. \( \frac{\pi}{6} \)
- 13. \( \frac{19\pi}{180°} \)

Find one angle with positive measure and one angle with negative measure coterminal with each angle. 14–16. Sample answers are given.

- 14. 60° 420°, −300°
- 15. 425° 785°, −295°
- 16. \( \frac{\pi}{3} \) \( \frac{7\pi}{3} \), \( -\frac{5\pi}{3} \)

Application ASTRONOMY For Exercises 17 and 18, use the following information.

Earth rotates on its axis once every 24 hours.

17. How long does it take Earth to rotate through an angle of 315°? 21 h
18. How long does it take Earth to rotate through an angle of \( \frac{\pi}{6} \)? 2 h

Practice and Apply

Draw an angle with the given measure in standard position. 19–26. See pp. 759A–759D.

- 19. 235°
- 20. 270°
- 21. 790°
- 22. 380°
- 23. −150°
- 24. −50°
- 25. \( \pi \)
- 26. \( -\frac{2\pi}{3} \)

DAILY INTERVENTION

Differentiated Instruction

Kinesthetic Have students work with a partner so that one person models an angle with outstretched arms or with two pencils or yardsticks. The other partner then names a positive and negative angle, less than or more than a full circle, that is coterminal with the modeled angle.
Rewrite each degree measure in radians and each radian measure in degrees.

27. 120° \( \frac{2\pi}{3} \) 
28. 60° \( \frac{\pi}{3} \) 
29. -15° \( -\frac{\pi}{12} \) 
30. -225° \( -\frac{5\pi}{4} \) 
31. 660° \( 11\pi \) 
32. 570° \( \frac{19\pi}{6} \) 
33. 158° \( \frac{79\pi}{9} \) 
34. 260° \( \frac{13\pi}{9} \) 
35. \( \frac{5\pi}{6} \) 
36. \( \frac{11\pi}{4} \) 
37. -\( \frac{\pi}{4} \) 
38. -\( \frac{\pi}{3} \) 
39. \( \frac{29\pi}{4} \) 
40. \( \frac{17\pi}{6} \) 
41. 9° 
42. 3°

Find one angle with positive measure and one angle with negative measure coterminal with each angle. 43–54. Sample answers are given.

43. 225°, 585°, -135° 
44. 30°, 390°, -330° 
45. -15°, 345°, -375° 
46. -140°, 220°, -500° 
47. 368°, 8°, -352° 
48. 760°, 400°, -320° 
49. \( \frac{3\pi}{4} \), \( \frac{11\pi}{5} \), \( \frac{5\pi}{4} \) 
50. \( \frac{7\pi}{6} \), \( \frac{19\pi}{12} \), \( \frac{5\pi}{6} \) 
51. -\( \frac{\pi}{4} \), \( \frac{3\pi}{4} \), \( \frac{\pi}{2} \) 
52. \( \frac{4\pi}{3} \), \( \frac{4\pi}{3} \), \( \frac{8\pi}{3} \) 
53. \( \frac{9\pi}{4} \), \( \frac{13\pi}{4} \), \( \frac{3\pi}{2} \) 
54. \( \frac{17\pi}{4} \), \( \frac{25\pi}{4} \), \( \frac{7\pi}{2} \)

### Driving

Some sport-utility vehicles (SUVs) use 15-inch radius wheels. When driven 40 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and nearest radian. 2689° per second; 47 radians per second

### Geometry

For Exercises 56 and 57, use the following information.

A sector is a region of a circle that is bounded by a central angle \( \theta \) and its intercepted arc. The area \( A \) of a sector with radius \( r \) and central angle \( \theta \) is given by \( A = \frac{1}{2}r^2\theta \), where \( \theta \) is measured in radians.

56. Find the area of a sector with a central angle of \( \frac{4\pi}{3} \) radians in a circle whose radius measures 10 inches. 209.4 in²
57. Find the area of a sector with a central angle of 150° in a circle whose radius measures 12 meters. about 188.5 m²

### Entertainment

Suppose the gondolas on the Navy Pier Ferris wheel were numbered from 1 through 40 consecutively in a counterclockwise fashion. If you were sitting in gondola number 3 and the wheel were to rotate counterclockwise through \( \frac{37\pi}{10} \) radians, which gondola used to be in the position that you are in now? number 17

### Cars

Use the Area of a Sector Formula in Exercises 56 and 57 to find the area swept by the rear windshield wiper of the car shown at the right. about 640.88 in²
How can angles be used to describe circular motion?

Angular velocity is defined by the equation

\[ \omega = \frac{\theta}{t} \]

where \( \theta \) is usually expressed in radians and \( t \) represents time. Find the angular velocity in radians per second of a point on a bicycle tire if it completes 2 revolutions in 3 seconds. D

### Mixed Review

Solve \( \triangle ABC \) by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-1)

64. \( a = 3.4 \), \( c = 6.0 \), \( \angle B = 56^\circ \)
65. \( B = 68^\circ \), \( b = 14.7 \)
66. \( B = 55^\circ \), \( c = 16 \)
67. \( a = 0.4 \), \( b = 0.4\sqrt{3} \)
68. \( A = 22^\circ \), \( a = 5.9 \), \( c = 15.9 \)

Find the margin of sampling error. (Lesson 12-9)

68. \( p = 72\% \), \( n = 100 \) about 8.98\%
69. \( p = 50\% \), \( n = 200 \) about 7.07\%

### Maintain Your Skills

#### Trigonometric Functions

#### Skills Practice, p. 783 and Practice, p. 784 (shown)

Draw an angle with the given measure in standard position.

1. \( 160^\circ \)
2. \( -75^\circ \)
3. \( -300^\circ \)

Rewrite each degree measure in radians and each radian measure in degrees.

4. \( 180^\circ \)
5. \( -270^\circ \)
6. \( -30^\circ \)

Find one angle with positive measure and one angle with negative measure coterminal with each angle. 33-36. Sample answers are given.

33. \( 60^\circ \) \( 120^\circ \) \( -120^\circ \)
34. \( 90^\circ \) \( 270^\circ \) \( -270^\circ \)
35. \( -15^\circ \) \( 165^\circ \) \(-45^\circ \)
36. \( 40^\circ \) \( 160^\circ \) \(-160^\circ \)

### Standardized Test Practice

#### Trigonometric Functions

#### Pre-Activity

How can angles be used to describe circular motion?

Read this introduction to Lesson 13.5 on the top of page 788 in your textbook.

If a pendulum swings through a complete revolution in 15 minutes, what is its angular velocity in degrees per minute? (Lesson 12-9)

#### Reading the Lesson

1. Make each degree measure with the corresponding radian measure on the right.

#### Standardized Test Practice

#### Reading to Learn Mathematics, p. 785

#### ELL

#### Mixed Review

#### Maintain Your Skills

#### Enrichment, p. 786

Making and Using a Hypsometer

A hypsometer is a device that can be used to measure the height of an object. To construct your own hypsometer, you will need a rectangular piece of heavy cardboard that is at least 7 in by 10 in, one straw, transparent tape, a string about 30 in long, and a small weight that can be attached to the string.

1. Mark off one increment along one side and one long side of the cardboard. Then attach the other side to the hypsometer by taping it to the side of the cardboard, as shown in the figure below. The diagram below shows how your hypsometer should look.
2. How can angles be used to describe circular motion?
Determine whether each situation involves a permutation or a combination. Then find the number of possibilities. (Lesson 12-2)

70. choosing an arrangement of 5 CDs from your 30 favorite CDs
71. choosing 3 different types of snack foods out of 7 at the store to take on a trip combination, 35

Find \( f \circ g(x) \) and \( (h \circ g)(x) \). (Lesson 7-7)

72. \( (g \circ h)(x) = 6x - 8 \), \( h \circ g(x) = 6x - 4 \)
73. \( (g \circ h)(x) = 4x^2 - 6x + 23 \), \( h \circ g(x) = 8x^2 + 34x + 44 \)

For Exercises 74 and 75, use the graph at the right.

The number of sports radio stations can be modeled by

\[ R(x) = 7.8x^2 + 16.6x + 95.8 \]

where \( x \) is the number of years since 1996. (Lesson 7-5)

74. Use synthetic substitution to estimate the number of sports radio stations for 2006. 1041.8
75. Evaluate \( R(12) \). What does this value represent? 1418.2 or about 1418; the number of sports radio stations in 2008

Open-Ended Assessment

Speaking Have students work in small groups to decide on an informal explanation of what a radian is and what coterminal angles are. Then have a reporter from each group share that explanation with the whole class.

Getting Ready for Lesson 13-3

PREREQUISITE SKILL Lesson 13-3 presents finding the trigonometric functions for general angles. Students will use their familiarity with rationalizing denominators as they find values of trigonometric functions. Exercises 76–81 should be used to determine your students’ familiarity with rationalizing denominators.

Assessment Options

Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 13-1 and 13-2. Lesson numbers are given to the right of exercises or instruction lines so students can review concepts not yet mastered.

Quiz (Lessons 13-1 and 13-2) is available on p. 831 of the Chapter 13 Resource Masters.

Answers

3. The graph at the right.

4. \( \sin \theta = \frac{10\sqrt{149}}{149}; \cos \theta = \frac{7\sqrt{149}}{149}; \tan \theta = \frac{10}{7}; \csc \theta = \frac{\sqrt{149}}{10}; \sec \theta = \frac{\sqrt{149}}{7}; \cot \theta = \frac{7}{10} \)

Lesson 13-2 Angles and Angle Measure 715

Online Lesson Plans

USA TODAY Education’s Online site offers resources and interactive features connected to each day’s newspaper. Experience TODAY, USA TODAY’s daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.
Getting Started

**Objective** Investigate measures in regular polygons using trigonometry.

**Materials**
- compass
- straightedge
- protractor

**Teach**
- Have students copy the figure and use a colored pencil to show which line is the apothem.
- Make sure the students draw circles with a radius (not a diameter) of 1 inch.
- If it is available, you may want to have students use computer software to draw the inscribed regular polygons.

**Assess**
In Exercises 1–3, students should
- be able to see the pattern in the table.
- understand how the apothem changes as the number of sides in the polygon increases.

In Exercises 4–7, students should
- be able to develop the formula for the apothem.
- understand the effect of the length of the radius on the formula.

### Collect the Data
- Use a compass to draw a circle with a radius of one inch. Inscribe an equilateral triangle inside of the circle. To do this, use a protractor to measure three angles of 120° at the center of the circle, since \( \frac{360°}{3} = 120° \). Then connect the points where the sides of the angles intersect the circle using a straightedge.
- The **apothem** of a regular polygon is a segment that is drawn from the center of the polygon perpendicular to a side of the polygon. Use the cosine of angle \( \theta \) to find the length of an apothem, labeled \( a \) in the diagram below.

### Analyze the Data
1. Make a table like the one shown below and record the length of the apothem of the equilateral triangle.

<table>
<thead>
<tr>
<th>Number of Sides, ( n )</th>
<th>( \theta )</th>
<th>( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>60</td>
<td>0.50</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>0.71</td>
</tr>
<tr>
<td>5</td>
<td>36</td>
<td>0.81</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>0.87</td>
</tr>
<tr>
<td>7</td>
<td>( \approx 26 )</td>
<td>0.90</td>
</tr>
<tr>
<td>8</td>
<td>22.5</td>
<td>0.92</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>0.94</td>
</tr>
<tr>
<td>10</td>
<td>18</td>
<td>0.95</td>
</tr>
</tbody>
</table>

In this table, the length of the apothem increases as the number of sides increases. The measure of \( \theta \) decreases.

2. What do you notice about the measure of \( \theta \) as the number of sides of the inscribed polygon increases? **The measure of \( \theta \) decreases.**

3. What do you notice about the values of \( a \)? **The length of the apothem increases as the number of sides increases.**

### Make a Conjecture
4. Suppose you inscribe a 20-sided regular polygon inside a circle. Find the measure of angle \( \theta \). \( 9° \)
5. Write a formula that gives the measure of angle \( \theta \) for a polygon with \( n \) sides. \( \theta = \frac{360°}{2n} \) or \( \theta = \frac{180°}{n} \)
6. Write a formula that gives the length of the apothem of a regular polygon inscribed in a circle of radius one inch. \( a = \cos \theta \)
7. How would the formula you wrote in Exercise 6 change if the radius of the circle was not one inch? See pp. 759A–759D.

### Resource Manager

**Teaching Algebra with Manipulatives**
- p. 23 (master for protractors)
- p. 301 (student recording sheet)

**Glencoe Mathematics Classroom Manipulative Kit**
- protractors
- rulers
- compasses
5-Minute Check Transparency 13-3 Use as a quiz or review of Lesson 13-2.

Mathematical Background notes are available for this lesson on p. 698C.

How can you model the position of riders on a skycoaster?

Ask students:

• What happens to the size of the swing (as measured by \(\theta\)) as the time after the plunge increases? The arc of the swing decreases, eventually to zero.  
• Is the \(t\) in the formula the same as the time that has elapsed since the rider plunged? No

TRIGONOMETRIC FUNCTIONS AND GENERAL ANGLES In Lesson 13-1, you found values of trigonometric functions whose domains were the set of all acute angles, angles between 0 and \(\frac{\pi}{2}\), of a right triangle. For \(t > 0\) in the equation above, you must find the cosine of an angle greater than \(\frac{\pi}{2}\). In this lesson, we will extend the domain of trigonometric functions to include angles of any measure.

Key Concept: Trigonometric Functions, \(\theta\) in Standard Position

Let \(\theta\) be an angle in standard position and let \(P(x, y)\) be a point on the terminal side of \(\theta\). Using the Pythagorean Theorem, the distance \(r\) from the origin to \(P\) is given by \(r = \sqrt{x^2 + y^2}\). The trigonometric functions of an angle in standard position may be defined as follows.

- \(\sin \theta = \frac{y}{r}\)  
- \(\cos \theta = \frac{x}{r}\)  
- \(\tan \theta = \frac{y}{x}, x \neq 0\)  
- \(\csc \theta = \frac{r}{y}, y \neq 0\)  
- \(\sec \theta = \frac{r}{x}, x \neq 0\)  
- \(\cot \theta = \frac{x}{y}, y \neq 0\)

Example 1: Evaluate Trigonometric Functions for a Given Point

Find the exact values of the six trigonometric functions of \(\theta\) if the terminal side of \(\theta\) contains the point \((5, -12)\).

From the coordinates given, you know that \(x = 5\) and \(y = -12\). Use the Pythagorean Theorem to find \(r\).

(continued on the next page)
1. Find the exact values of the six trigonometric functions of \( \theta \) if the terminal side of \( \theta \) contains the point (8, -15).

\[
\sin \theta = -\frac{15}{17}; \cos \theta = \frac{8}{17}; \\
\tan \theta = -\frac{15}{8}; \csc \theta = -\frac{17}{15}; \\
\sec \theta = \frac{17}{8}; \cot \theta = -\frac{8}{15}.
\]

**Teaching Tip:** When finding functions for an angle, students may find it helpful to sketch an angle in standard position and drop a perpendicular to the \( x \)-axis to form a right triangle. Then they can see that, as the angle increases from 90° to 180°, the \( y \) value approaches 0, and the values of the other two sides approach each other.

2. Find the values of the six trigonometric functions for an angle in standard position that measures 180°.

\[
\sin \theta = 0; \cos \theta = -1; \\
\tan \theta = 0; \csc \theta \text{ is undefined}; \\
\sec \theta = -1; \cot \theta \text{ is undefined}.
\]

If the terminal side of angle \( \theta \) lies on one of the axes, \( \theta \) is called a **quadrantal angle**. The quadrantal angles are 0°, 90°, 180°, and 270°. Notice that for these angles either \( x \) or \( y \) is equal to 0. Since division by zero is undefined, two of the trigonometric values are undefined for each quadrantal angle.

### Key Concept

**Quadrantal Angles**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \theta = 0^\circ ) or 0 radians</th>
<th>( \theta = 90^\circ ) or ( \frac{\pi}{2} ) radians</th>
<th>( \theta = 180^\circ ) or ( \pi ) radians</th>
<th>( \theta = 270^\circ ) or ( \frac{3\pi}{2} ) radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>undefined</td>
<td>0</td>
<td>undefined</td>
<td>undefined</td>
</tr>
<tr>
<td>( \csc \theta )</td>
<td>undefined</td>
<td>( \frac{1}{0} ) or undefined</td>
<td>undefined</td>
<td>undefined</td>
</tr>
<tr>
<td>( \sec \theta )</td>
<td>( \frac{1}{0} ) or undefined</td>
<td>undefined</td>
<td>undefined</td>
<td>undefined</td>
</tr>
<tr>
<td>( \cot \theta )</td>
<td>undefined</td>
<td>undefined</td>
<td>undefined</td>
<td>undefined</td>
</tr>
</tbody>
</table>

**REFERENCE ANGLES**

To find the values of trigonometric functions of angles greater than 90° (or less than 0°), you need to know how to find the measures of reference angles. If \( \theta \) is a nonquadrantal angle in standard position, its **reference angle**, \( \theta' \), is defined as the acute angle formed by the terminal side of \( \theta \) and the \( x \)-axis.

**Unlocking Misconceptions**

In Example 1, watch for students who think that \( r \) must be a negative number any time either \( x \) or \( y \) has a negative value.
You can use the rule below to find the reference angle for any nonquadrantal angle $\theta$ where $0^\circ < \theta < 360^\circ$ (or $0 < \theta < 2\pi$).

**Key Concept**

**Reference Angle Rule**

For any nonquadrantal angle $\theta$, $0^\circ < \theta < 360^\circ$ (or $0 < \theta < 2\pi$), its reference angle $\theta'$ is defined as follows:

- **Quadrant I**
  - $\theta' = \theta$

- **Quadrant II**
  - $\theta' = 180^\circ - \theta$
  - ($\theta' = \pi - \theta$)

- **Quadrant III**
  - $\theta' = \theta - 180^\circ$
  - ($\theta' = \theta - \pi$)

- **Quadrant IV**
  - $\theta' = 360^\circ - \theta$
  - ($\theta' = 2\pi - \theta$)

If the measure of $\theta$ is greater than $360^\circ$ or less than $0^\circ$, its reference angle can be found by associating it with a coterminal angle of positive measure between $0^\circ$ and $360^\circ$.

**Example 3**  
**Find the Reference Angle for a Given Angle**

Sketch each angle. Then find its reference angle.

a. $300^\circ$

Because the terminal side of $300^\circ$ lies in Quadrant IV, the reference angle is $360^\circ - 300^\circ$ or $60^\circ$.

b. $-\frac{2\pi}{3}$

A coterminal angle of $-\frac{2\pi}{3}$ is $2\pi - \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Because the terminal side of this angle lies in Quadrant III, the reference angle is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.

To use the reference angle $\theta'$ to find a trigonometric value of $\theta$, you need to know the sign of that function for an angle $\theta$. From the function definitions, these signs are determined by $x$ and $y$, since $r$ is always positive. Thus, the sign of each trigonometric function is determined by the quadrant in which the terminal side of $\theta$ lies.

The chart below summarizes the signs of the trigonometric functions for each quadrant.

<table>
<thead>
<tr>
<th>Function</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$ or $\csc \theta$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\cos \theta$ or $\sec \theta$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$\tan \theta$ or $\cot \theta$</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

**TEACHING TIP**

For example, because $\sin \theta = \frac{y}{r}$ and $r$ is always positive, $\sin \theta$ is positive wherever $y > 0$, which is in Quadrants I and II.

For more information, visit [www.algebra2.com/extra_examples](www.algebra2.com/extra_examples)

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**Teacher to Teacher**

David S. Daniels  
Longmeadow H.S., Longmeadow, MA

“Have students measure the distance from the tip of the minute hand on the classroom clock to the ceiling at 5 minute or 30° intervals, starting at the top of an hour. Then have students plot the data and describe the graph.

Students will discover that the distance from the ceiling is a function of the number of degrees through which the minute hand has turned.”

---

**Reference Angles**

- **In-Class Example**
  - Sketch each angle. Then find its reference angle.
  - a. $330^\circ$
  - b. $-\frac{5\pi}{6}$
In-Class Examples

4. Find the exact value of each trigonometric function.
   a. \( \sin 135^\circ = \frac{\sqrt{2}}{2} \)
   b. \( \cot \frac{7\pi}{3} = \frac{\sqrt{3}}{3} \)

5. Suppose \( \theta \) is an angle in standard position whose terminal side is in Quadrant III and \( \csc \theta = -\frac{5}{3} \). Find the exact values of the remaining five trigonometric functions of \( \theta \).
   \[
   \sin \theta = -\frac{3}{5}, \cos \theta = -\frac{4}{5}, \tan \theta = \frac{3}{4}, \sec \theta = -\frac{5}{4}, \cot \theta = -\frac{4}{3}
   \]

Study Tip

Look Back
To review trigonometric values of angles measuring 30°, 45°, and 60°, see Lesson 13-1.

Example 4 Use a Reference Angle to Find a Trigonometric Value

Find the exact value of each trigonometric function.

a. \( \sin 120^\circ \)
   Because the terminal side of \( 120^\circ \) lies in Quadrant II, the reference angle \( \theta' \) is \( 180^\circ - 120^\circ = 60^\circ \). The sine function is positive in Quadrant II, so \( \sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2} \).

b. \( \cot \frac{7\pi}{4} \)
   Because the terminal side of \( \frac{7\pi}{4} \) lies in Quadrant IV, the reference angle \( \theta' \) is \( 2\pi - \frac{7\pi}{4} = \frac{\pi}{4} \). The cotangent function is negative in Quadrant IV.
   \[
   \cot \frac{7\pi}{4} = -\cot \frac{\pi}{4}
   = -\cot 45^\circ = -1
   \]
   \[
   \cot 45^\circ = 1
   \]

If you know the quadrant that contains the terminal side of \( \theta \) in standard position and the exact value of one trigonometric function of \( \theta \), you can find the values of the other trigonometric functions of \( \theta \) using the function definitions.

Example 5 Quadrant and One Trigonometric Value of \( \theta \)

Suppose \( \theta \) is an angle in standard position whose terminal side is in the Quadrant III and \( \sec \theta = -\frac{4}{3} \). Find the exact values of the remaining five trigonometric functions of \( \theta \).

Draw a diagram of this angle, labeling a point \( P(x, y) \) on the terminal side of \( \theta \). Use the definition of secant to find the values of \( x \) and \( r \).

\[
\sec \theta = -\frac{4}{3} \quad \text{Given}
\]
\[
\frac{r}{x} = -\frac{4}{3} \quad \text{Definition of secant}
\]

Since \( x \) is negative in Quadrant III and \( r \) is always positive, \( x = -3 \) and \( r = 4 \). Use these values and the Pythagorean Theorem to find \( y \).

Daily Intervention

Differentiated Instruction

Auditory/Musical Have students work in small groups to create a jingle, song, rap, or short poem to help them remember the basic equivalences between angle measures in degrees and radians.
Lesson 13-3  Trigonometric Functions of General Angles  721

### In-Class Example

#### Example 6  Find Coordinates Given a Radius and an Angle

**ROBOTICS** Use the figure for Example 6 in the Student Edition to find the new position of the object relative to the pivot point for a robotic arm that is 3 meters long and that rotates through an angle of 150°.

\[
\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) \text{ or about 2.60 meters to the left of } O \text { and 1.5 meters in front of } O
\]

Let the position of point \(A\) be coordinates \((x, y)\). Then, use the definitions of sine and cosine to find the value of \(x\) and \(y\). The value of \(r\) is the length of the robotic arm, 4 meters. Because \(B\) is in Quadrant II, the cosine of 135° is negative.

- \(\cos 135° = \frac{x}{r}\) or \(x = r \cos 135° = -\frac{
\sqrt{2}}{2}\times 3 = \frac{-3\sqrt{2}}{2}\)
- \(\sin 135° = \frac{y}{r}\) or \(y = r \sin 135° = \frac{\sqrt{2}}{2}\times 3 = \frac{3\sqrt{2}}{2}\)

The exact coordinates of \(B\) are \((-2\sqrt{2}, 2\sqrt{2})\). Since \(2\sqrt{2}\) is about 2.82, the object is about 2.82 meters to the left of the pivot point and about 2.82 meters in front of the pivot point.

Just as an exact point on the terminal side of an angle can be used to find trigonometric function values, trigonometric function values can be used to find the exact coordinates of a point on the terminal side of an angle.
3. To find the value of a trigonometric function of \( \theta \), where \( \theta \) is greater than 90°, find the value of the trigonometric function for \( \theta' \), then use the quadrant in which the terminal side of \( \theta' \) lies to determine the sign of the trigonometric function value of \( \theta \).

4. \( \sin \theta = \frac{8}{17}, \cos \theta = -\frac{15}{17}, \tan \theta = -\frac{8}{15}, \csc \theta = \frac{17}{8}, \sec \theta = -\frac{17}{15}, \cot \theta = -\frac{15}{8} \)

5. \( \sin \theta = 0, \cos \theta = -1, \tan \theta = 0, \csc \theta = \text{undefined}, \sec \theta = -1, \cot \theta = \text{undefined} \)

6. \( \sin \theta = \frac{\sqrt{2}}{2}, \cos \theta = \frac{\sqrt{2}}{2}, \tan \theta = 1, \csc \theta = \sqrt{2}, \sec \theta = \sqrt{2}, \cot \theta = 1 \)

14. \( \sin \theta = \frac{\sqrt{3}}{2}, \tan \theta = -\sqrt{3}, \csc \theta = \frac{2\sqrt{3}}{3}, \sec \theta = -2, \cot \theta = -\frac{\sqrt{3}}{3} \)

15. \( \sin \theta = -\frac{\sqrt{3}}{3}, \cos \theta = -\frac{\sqrt{3}}{3}, \tan \theta = -\sqrt{2}, \csc \theta = -\frac{\sqrt{6}}{2}, \sec \theta = \sqrt{3} \)

Find the exact values of the six trigonometric functions of \( \theta \) if the terminal side of \( \theta \) in standard position contains the given point. 4–6. See margin.

4. (−15, 8)  5. (−3, 0)  6. (4, 4)

Sketch each angle. Then find its reference angle.

7. 235°  8. \( \frac{7\pi}{4} \)  9. −240° 60°


Find the exact value of each trigonometric function.

10. \( \sin 300° = -\frac{\sqrt{3}}{2} \)  11. \( \cos 180° = -1 \)

12. \( \tan \frac{5\pi}{3} = -\sqrt{3} \)  13. \( \sec \frac{7\pi}{6} = -\frac{2\sqrt{3}}{3} \)

Suppose \( \theta \) is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of \( \theta \). 14–15. See margin.

14. \( \cos \theta = -\frac{1}{2} \), Quadrant II  15. \( \cot \theta = -\frac{\sqrt{3}}{2} \), Quadrant IV

Application 16. BASKETBALL The maximum height \( H \) in feet that a basketball reaches after being shot is given by the formula \( H = \frac{V_0^2}{64} \sin^2 \theta \), where \( V_0 \) represents the initial velocity in feet per second, \( \theta \) represents the degree measure of the angle that the path of the basketball makes with the ground. Find the maximum height reached by a ball shot with an initial velocity of 30 feet per second at an angle of 70°. about 12.4 ft
37. undefined
38. $\sin \theta = -\frac{4}{5}, \tan \theta = -\frac{4}{3}, \csc \theta = -\frac{5}{4}, \sec \theta = -\frac{5}{3}, \cot \theta = -\frac{3}{4}$

45. **SKYCOASTING** Mikhail and Anya visit a local amusement park to ride a skycoaster. After the first several swings, the angle the skycoaster makes with the vertical is modeled by $\theta = 0.2 \cos \pi t$, with $\theta$ measured in radians and $t$ measured in seconds. Determine the measure of the angle for $t = 0, 0.5, 1, 1.5, 2, 2.5$, and $3$ in both radians and degrees. $0.2, 0, -0.2, 0, 0.2, 0$, and $-0.2$; or about $11.5^\circ, 0^\circ, -11.5^\circ, 0^\circ, 11.5^\circ, 0^\circ$, and $-11.5^\circ$.

46. **NAVIGATION** Ships and airplanes measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 20^\circ$, where $\theta$ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is $60^\circ$. 6092.5 ft

Suppose $\theta$ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of $\theta$. 50–52. See pp. 759A–759D.

47. $\cos \theta = \frac{3}{5}$, Quadrant IV.
48. $\tan \theta = -\frac{1}{5}$, Quadrant II.
49. $\sin \theta = \frac{1}{3}$, Quadrant II.
50. $\cot \theta = \frac{2}{1}$, Quadrant III.
51. $\sec \theta = -\sqrt{10}$, Quadrant III.
52. $\csc \theta = -\frac{5}{2}$, Quadrant IV.

**BASEBALL** For Exercises 53 and 54, use the following information.

The formula $R = \frac{V_y}{g}$ gives the distance that a ball is hit that is hit at an initial velocity of $V_y$ feet per second at an angle of $\theta$ with the ground.

53. If the ball was hit with an initial velocity of 80 feet per second at an angle of $30^\circ$, how far was it hit? **about 173.2 ft**

54. Which angle will result in the greatest distance? Explain your reasoning. **$45^\circ$; $2 \times 45^\circ$ or $90^\circ$ yields the greatest value for $sin 20^\circ$**

55. **CAROUSELS** Anthony’s little brother gets on a carousel that is 8 meters in diameter. At the start of the ride, his brother is 3 meters from the fence to the ride. How far will his brother be from the fence after the carousel rotates $240^\circ$?

9 meters

**Online Research Data Update** What is the diameter of the world’s largest carousel? Visit www.algebra2.com/data_update to learn more.

**CRITICAL THINKING** Suppose $\theta$ is an angle in standard position with the given conditions. State the quadrant(s) in which the terminal side of $\theta$ lies.

56. $\sin \theta > 0, \cos \theta < 0$, I, II
57. $\sin \theta > 0, \cos \theta < 0$, I, III
58. $\tan \theta > 0, \cos \theta < 0$, III

Lesson 13-3 Trigonometric Functions of General Angles 723
Open-Ended Assessment

Writing Have students choose an angle measure, draw an angle with that measure in standard position, find the reference angle, and give the values of all 6 of the trigonometric functions, in both degrees and radians. Then display some of these in the classroom.

Getting Ready for Lesson 13-4

PREREQUISITE SKILL Lesson 13-4 presents the Law of Sines which will require students to solve equations involving trigonometric functions as they apply the Law of Sines. Exercises 72–77 should be used to determine your students’ familiarity with solving equations with trigonometric functions.

Maintain Your Skills

Rewrite each degree measure in radians and each radian measure in degrees.

\[ 62. \quad 90^\circ = \frac{\pi}{2} \quad 63. \quad \frac{5\pi}{3} = 300^\circ \quad 64. \quad 5 \cdot \frac{\pi}{\pi} = 286.5^\circ \]

Write an equation involving sin, cos, or tan that can be used to find \( x \). Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

\[ 65. \quad \sin 28^\circ = \frac{12}{13}, \quad 5.6 \quad 66. \quad \cos 43^\circ = \frac{8}{13}, \quad 60.7 \quad 67. \quad \sin x^\circ = \frac{5}{13}, \quad 23 \]

Use Cramer’s Rule to solve each system of equations.

\[ 69. \quad 3x - 4y = 13 \quad (7, 2) \quad 70. \quad 5x + 7y = 1 \quad (-4, 3) \quad 71. \quad 2x + 3y = -2 \quad (5, -4) \]

\[ -2x + 5y = -4 \quad 3x + 5y = 3 \quad -6x + y = -34 \]

Getting Ready for the Next Lesson

PREREQUISITE SKILL Solve each equation. Round to the nearest tenth.

\[ 72. \quad \frac{a}{\sin A^\circ} = \frac{8}{\sin 65^\circ} = 4.7 \quad 73. \quad \frac{b}{\sin 45^\circ} = \frac{21}{\sin 10^\circ} = 15.1 \quad 74. \quad \frac{c}{\sin 60^\circ} = \frac{3}{\sin 75^\circ} = 2.7 \]

\[ 75. \quad \sin \frac{A^\circ}{14} = \sin 104^\circ \quad 76. \quad \frac{b}{\sin R^\circ} = \frac{55^\circ}{7} = 20.6^\circ \quad 77. \quad \frac{c}{\sin C^\circ} = \frac{35^\circ}{9} = 39.6^\circ \]

724 Chapter 13 Trigonometric Functions
5-Minute Check Transparency 13-4 Use as a quiz or review of Lesson 13-3.

Mathematical Background notes are available for this lesson on p. 698D.

How can trigonometry be used to find the area of a triangle?
Ask students:
• What kind of triangles does the height separate the triangle into? right triangles
• Does the size of angle C make any difference in this derivation of the area formula? no

LAW OF SINES You can find two other formulas for the area of the triangle above in a similar way. These formulas, summarized below, allow you to find the area of any triangle when you know the measures of two sides and the included angle.

Key Concept

Area of a Triangle

Words The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.

Symbols

area = \( \frac{1}{2}bc \sin A \)
area = \( \frac{1}{2}ac \sin B \)
area = \( \frac{1}{2}ab \sin C \)

Example 1 Find the Area of a Triangle

Find the area of \( \triangle ABC \) to the nearest tenth.
In this triangle, \( a = 5 \), \( c = 6 \), and \( B = 112^\circ \).
Choose the second formula because you know the values of its variables.

Area = \( \frac{1}{2}ac \sin B \)
Area formula
= \( \frac{1}{2}(5)(6) \sin 112^\circ \)
Replace \( a \) with 5, \( c \) with 6, and \( B \) with 112°.
≈ 13.9 Use a calculator.

To the nearest tenth, the area is 13.9 square feet.
LAW OF SINES

**In-Class Examples**

**Teaching Tip** In Example 1, ask students if they knew only the values for sides \( c \) and \( a \), and for angle \( A \), could they use the formula to find the area of the triangle? No, to use the formula, the angle must be included in the known sides.

**Example 1** Find the area of \( \triangle ABC \) to the nearest tenth. \( 3.8 \text{ cm}^2 \)

![Diagram: \( \triangle ABC \) with sides 6 cm, 3 cm, and angle 25°]

**Example 2** Solve \( \triangle ABC \).

![Diagram: \( \triangle ABC \) with angles 100°, 53°, and 27°, and side 9 cm]

\( A = 27°, a \approx 5.1, c \approx 11.1 \)

---

**Study Tip**

**Alternate Representations**

The Law of Sines may also be written as

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

**Key Concept**

**Law of Sines**

Let \( \triangle ABC \) be any triangle with \( a, b \), and \( c \) representing the measures of sides opposite angles with measurements \( A, B, \) and \( C \) respectively. Then,

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

The Law of Sines can be used to write three different equations.

\[
\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}
\]

In Lesson 13-1, you learned how to solve right triangles. To solve any triangle, you can apply the Law of Sines if you know

- the measures of two angles and any side or
- the measures of two sides and the angle opposite one of them.

**Example 2** Solve a Triangle Given Two Angles and a Side

Solve \( \triangle ABC \).

You are given the measures of two angles and a side. First, find the measure of the third angle.

\[
45° + 55° + B = 180°
\]

The sum of the angle measures of a triangle is 180°.

\[
B = 80° \quad 180° - (45° + 55°) = 80°
\]

Now use the Law of Sines to find \( a \) and \( b \). Write two equations, each with one variable.

\[
\frac{\sin A}{a} = \frac{\sin C}{c}
\]

Replace \( A \) with 45°, \( B \) with 80°, \( C \) with 55°, and \( c \) with 12.

\[
\sin A = \frac{\sin 55°}{12}
\]

Replace \( A \) with 45°, \( B \) with 80°, \( C \) with 55°, and \( c \) with 12.

\[
\sin 45° = \frac{\sin 55°}{a}
\]

Solve for the variable.

\[
a = 10.4 \quad \text{Use a calculator.}
\]

Therefore, \( B = 80°, a \approx 10.4, \) and \( b = 14.4 \).

---

**Unlocking Misconceptions**

Students may think the Law of Sines only works for right triangles. Clarify that this formula works for any triangle, as does the Law of Cosines explored in Lesson 13-5.
When solving a triangle, you must analyze the data you are given to determine whether there is a solution. For example, if you are given the measures of two angles and a side, as in Example 2, the triangle has a unique solution. However, if you are given the measures of two sides and the angle opposite one of them, a single solution may not exist. One of the following will be true.

- No triangle exists, and there is no solution.
- Exactly one triangle exists, and there is one solution.
- Two triangles exist, and there are two solutions.

### Example 3: One Solution

In \( \triangle ABC \), \( A = 118^\circ \), \( a = 20 \), and \( b = 17 \). Determine whether \( \triangle ABC \) has no solution, one solution, or two solutions. Then solve \( \triangle ABC \).

Because angle \( A \) is obtuse and \( a > b \), you know that one solution exists.

Make a sketch and then use the Law of Sines to find \( B \).

\[
\sin B = \frac{\sin 118^\circ}{20} \quad \text{Law of Sines}
\]

\[
\sin B = \frac{17 \sin 118^\circ}{20} \quad \text{Multiply each side by 17.}
\]

\[
\sin B = 0.7505 \quad \text{Use a calculator.}
\]

\[
B = 49^\circ \quad \text{Use the \( \sin^{-1} \) function.}
\]

The measure of angle \( C \) is approximately \( 180^\circ - (118^\circ + 49^\circ) \) or \( 13^\circ \).

Use the Law of Sines again to find \( c \).

\[
\sin \frac{13}{c} = \sin \frac{118^\circ}{20} \quad \text{Law of Sines}
\]

\[
c = \frac{20 \sin 13^\circ}{\sin 118^\circ} \quad \text{or about 5.1} \quad \text{Use a calculator.}
\]

Therefore, \( B = 49^\circ \), \( C = 13^\circ \), and \( c \approx 5.1 \).
4 In \( \triangle ABC \), \( A = 125^\circ \), \( a = 35 \), and \( b = 32 \). Determine whether \( \triangle ABC \) has no solution, one solution, or two solutions. Then solve \( \triangle ABC \).**no**

5 In \( \triangle ABC \), \( A = 25^\circ \), \( a = 5 \), and \( b = 10 \). Determine whether \( \triangle ABC \) has no solution, one solution, or two solutions. Then solve \( \triangle ABC \).**two:** \( B \approx 58^\circ \), \( C \approx 97^\circ \), \( c \approx 11.7 \); \( B \approx 122^\circ \), \( C \approx 33^\circ \), \( c \approx 6.4 \)

---

**Example 4** No Solution

In \( \triangle ABC \), \( A = 50^\circ \), \( a = 5 \), and \( b = 9 \). Determine whether \( \triangle ABC \) has no solution, one solution, or two solutions. Then solve \( \triangle ABC \).

Since angle \( A \) is acute, find \( b \) sin \( A \) and compare it with \( a \).

\[
b \sin A = 9 \sin 50^\circ \\
= 6.9
\]

Use a calculator.

Since \( 5 < 6.9 \), there is no solution.

---

When two solutions for a triangle exist, it is called the ambiguous case.

**Example 5** Two Solutions

In \( \triangle ABC \), \( A = 39^\circ \), \( a = 10 \), and \( b = 14 \). Determine whether \( \triangle ABC \) has no solution, one solution, or two solutions. Then solve \( \triangle ABC \).

Since angle \( A \) is acute, find \( b \) sin \( A \) and compare it with \( a \).

\[
b \sin A = 14 \sin 39^\circ \\
= 8.81
\]

Use a calculator.

Since \( 14 > 10 > 8.81 \), there are two solutions. Thus, there are two possible triangles to be solved.

**Case 1** Acute Angle \( B \)

First, use the Law of Sines to find \( B \).

\[
\begin{align*}
\sin B &= \frac{\sin 39^\circ}{10} \\
\sin B &= \frac{14 \sin 39^\circ}{10} \\
\sin B &= 0.8810 \\
B &= 62^\circ
\end{align*}
\]

The measure of angle \( C \) is approximately \( 180^\circ - (39^\circ + 62^\circ) \) or \( 79^\circ \).

Use the Law of Sines again to find \( c \).

\[
\begin{align*}
\sin 79^\circ &= \frac{\sin 39^\circ}{10} \\
c &= \frac{10 \sin 79^\circ}{\sin 39^\circ} \\
c &\approx 15.6
\end{align*}
\]

Therefore, \( B \approx 62^\circ \), \( C \approx 79^\circ \), and \( c \approx 15.6 \).

**Case 2** Obtuse Angle \( B \)

To find \( B \), you need to find an obtuse angle whose sine is also 0.8810. To do this, subtract the angle given by your calculator, 62°, from 180°. So \( B \) is approximately 180° − 62° or 118°.

The measure of angle \( C \) is approximately 180° − (39° + 118°) or 23°.

Use the Law of Sines to find \( c \).

\[
\begin{align*}
\sin 23^\circ &= \frac{\sin 39^\circ}{10} \\
c &= \frac{10 \sin 23^\circ}{\sin 39^\circ} \\
c &\approx 6.2
\end{align*}
\]

Therefore, \( B \approx 118^\circ \), \( C \approx 23^\circ \), and \( c \approx 6.2 \).

---

**Differentiated Instruction**

**Intrapersonal** Have students write a journal entry about which example they found the most challenging and why. Ask them to include any questions they still have about the lesson.
Example 6 Use the Law of Sines to Solve a Problem

LIGHTHOUSES A lighthouse is located on a rock at a certain distance from a straight shore. The light revolves counterclockwise at a steady rate of one revolution per minute. As the beam revolves, it strikes a point on the shore that is 2000 feet from the lighthouse. Three seconds later, the light strikes a point 750 feet further down the shore. To the nearest foot, how far is the lighthouse from the shore?

Because the lighthouse makes one revolution every 60 seconds, the angle through which the light revolves in 3 seconds is \( \frac{3}{60}(360^\circ) = 18^\circ \).

Use the Law of Sines to find the measure of angle \( \alpha \).

\[
\sin \alpha = \frac{\sin 18^\circ}{2000} = \frac{\sin 18^\circ}{750}
\]

Multiply each side by 2000.

Use a calculator.

\( \alpha = 55^\circ \) Use the \( \sin^{-1} \) function.

Use this angle measure to find the measure of angle \( \theta \). Since \( \triangle ABC \) is a right triangle, the measures of angle \( \alpha \) and \( \angle BAC \) are complementary.

\( \alpha + m \angle BAC = 90^\circ \) Angles \( \alpha \) and \( \angle BAC \) are complementary.

\( 55^\circ + (\theta + 18^\circ) = 90^\circ \) \( a = 55^\circ \) and \( m \angle BAC = \theta + 18^\circ \)

\( \theta + 73^\circ = 90^\circ \) Simplify.

\( \theta = 17^\circ \) Solve for \( \theta \).

To find the distance from the lighthouse to the shore, solve \( \triangle ABD \) for \( d \).

\[
\cos \theta = \frac{AB}{AD}
\]

Cosine ratio

\[
\cos 17^\circ = \frac{d}{2000}
\]

\( \theta = 17^\circ \) and \( AD = 2000 \)

\( d = 2000 \cos 17^\circ \) Solve for \( d \).

\( d = 1913 \) Use a calculator.

The distance from the lighthouse to the shore, to the nearest foot, is 1913 feet. This answer is reasonable since 1913 is less than 2000.

Check for Understanding

Concept Check

1. Determine whether the following statement is sometimes, always or never true. Explain your reasoning.
   If given the measure of two sides of a triangle and the angle opposite one of them, you will be able to find a unique solution.

2. OPEN ENDED Give an example of a triangle that has two solutions by listing measures for \( A, a, \) and \( b \), where \( a \) and \( b \) are in centimeters. Then draw both cases using a ruler and protractor. Sample answer: \( A = 42^\circ, a = 2.6 \text{ cm}, b = 3.2 \text{ cm}; \) See margin for drawings.

In-Class Example

6 Lighthouses Refer to Example 6 in the Student Edition. Suppose a different lighthouse has a beam that revolves at the same rate (one revolution per minute) but the beam strikes a point on the shore that is 1840 feet from the lighthouse. Two seconds later, the light strikes a point 500 feet farther down the shore. To the nearest foot, how far is the lighthouse from the shore? 1625 ft

Practice/Apply

3

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 13.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Answer

2.

\[
\begin{align*}
&3.2 \text{ cm} \\
&2.6 \text{ cm} \\
\end{align*}
\]

\[
\begin{align*}
&3.9 \text{ cm} \\
&2.6 \text{ cm} \\
\end{align*}
\]

Lesson 13-4 Law of Sines 729
DAILY INTERVENTION FIND THE ERROR
Have students sketch a triangle and label sides a, b, and angle A on the sketch. Make sure students label angle A opposite side a.

About the Exercises...
Organization by Objective
- One, Two, or No Solutions: 28–37

Odd/Even Assignments
Exercises 14–37 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide
Basic: 15–37 odd, 42–58
Average: 15–37 odd, 42–58
Advanced: 14–40 even, 41–54 (optional: 55–58)

Answers
3. The information given is of two sides and an angle, but the angle is not between the two sides, therefore the area formula involving sine cannot be used.
6. C = 30°, a = 2.9, c = 1.5
7. B = 80°, a = 32.0, b = 32.6
8. B = 20°, A = 20°, a = 20.2

3. FIND THE ERROR Dulce and Gabe are finding the area of \( \triangle ABC \) for \( A = 64° \), \( a = 15 \) meters, and \( b = 8 \) meters using the sine function.

\[
\text{Dulce} \quad \text{Area} = \frac{1}{2} (15)(8) \sin 64° = 53.9 \text{ m}^2
\]

\[
\text{Gabe} \quad \text{There is not enough information to find the area of } \triangle ABC.
\]

Who is correct? Explain your reasoning. Gabe; see margin for explanation.

Guided Practice

Find the area of \( \triangle ABC \) to the nearest tenth.

4. 10.5 in²

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. 6–8. See margin.

6. 7. 8.

Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

9. \( A = 123°, a = 12, b = 23 \) no

10. \( A = 30°, a = 3, b = 4 \) no

11. \( A = 55°, a = 10, b = 5 \) one; \( B \approx 24°, C \approx 101°, c \approx 12.0 \)

12. \( A = 145°, a = 18, b = 10 \) one; \( B \approx 19°, C \approx 16°, c \approx 8.7 \)

Application

WOODWORKING Latisha is constructing a triangular brace from three beams of wood. She is to join the 6-meter beam to the 7-meter beam so that angle opposite the 7-meter beam measures 75°. To what length should Latisha cut the third beam in order to form the triangular brace? Round to the nearest tenth. 5.5 m

Practice and Apply

Find the area of \( \triangle ABC \) to the nearest tenth.

14. 15. 19.5 yd²

16. \( B = 85°, c = 23 \text{ ft}, a = 50 \text{ ft} \) 572.8 ft²

17. \( A = 60°, b = 12 \text{ cm}, c = 12 \text{ cm} \) 62.4 cm²

18. \( C = 136°, a = 3 \text{ m}, b = 4 \text{ m} \) 4.2 m²

28. no
29. one; \( B \approx 36°, C \approx 45°, c \approx 1.8 \)
30. two; \( B \approx 72°, C \approx 75°, c \approx 3.5 \);
   \( B \approx 108°, C \approx 39°, c \approx 2.3 \)
31. no
32. one; \( B = 90°, C = 60°, c = 24.2 \)
33. one; \( B \approx 18°, C \approx 101°, c \approx 25.8 \)
34. two; \( B \approx 56°, C = 72°, c = 229.3 \);
   \( B \approx 124°, C = 4°, c = 16.8 \)
35. two; \( B = 85°, C = 15°, c = 2.4 \);
   \( B = 95°, C = 5°, c = 0.8 \)
36. one; \( B = 23°, C = 129°, c = 14.1 \)
37. two; \( B = 65°, C = 68°, c = 84.9 \);
   \( B = 115°, C = 18°, c = 28.3 \)

Extra Practice

See page 858.
Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

20. \( \angle B = 62^\circ, \angle C = 17^\circ, \angle A = 101^\circ \)
   \( a = 55.6, b = 48.2, c = 3.4 \)

21. \( \angle A = 73^\circ, \angle B = 21^\circ, \angle C = 37^\circ \)
   \( a = 55.6, b = 48.2, c = 3.4 \)

22. \( \angle A = 40^\circ, \angle B = 65^\circ, \angle C = 2.8 \)
   \( a = 55.6, b = 48.2, c = 3.4 \)

23. \( \angle A = 47^\circ, \angle B = 68^\circ, \angle C = 5.1 \)
   \( a = 55.6, b = 48.2, c = 3.4 \)

24. \( \angle A = 97^\circ, \angle B = 20^\circ, \angle C = 75^\circ \)
   \( a = 55.6, b = 48.2, c = 3.4 \)

25. \( \angle A = 50^\circ, \angle B = 25^\circ, \angle C = 13^\circ \)
   \( a = 55.6, b = 48.2, c = 3.4 \)

26. \( A = 50^\circ, a = 2.5, c = 3 \)
   \( C = 67^\circ, B = 63^\circ, b = 2.9 \)

Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. 28–37. See margin.

28. \( A = 124^\circ, a = 1, b = 2 \)
   \( 29. A = 99^\circ, a = 2.5, b = 1.5 \)

30. \( A = 33^\circ, a = 2, b = 3.5 \)
   \( 31. A = 68^\circ, a = 3, b = 5 \)

32. \( A = 30^\circ, a = 14, b = 28 \)
   \( 33. A = 61^\circ, a = 23, b = 8 \)

34. \( A = 52^\circ, a = 190, b = 200 \)
   \( 35. A = 80^\circ, a = 9, b = 9.1 \)

36. \( A = 28^\circ, a = 8.5, b = 7.2 \)
   \( 37. A = 47^\circ, a = 67, b = 83 \)

38. RADIO A radio station providing local tourist information has its transmitter on Beacon Road, 8 miles from where it intersects with the interstate highway. If the radio station has a range of 5 miles, between what two distances from the intersection can cars on the interstate turn in to hear this information? 4.6 and 8.5 mi

39. FOREST Two forest Rangers, 12 miles from each other on a straight road, both sight an illegal bonfire away from the road. Using their radios to communicate with each other, they determine that the fire is between them. The first ranger’s line of sight from the fire makes a 48° angle of depression, with the road, and the second ranger’s line of sight to the fire makes a 63° angle with the road. How far is the fire from each ranger? 7.5 mi from Ranger B, 10.9 mi from Ranger A

40. BALLOONING As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts the angles of depression are 64° and 7°. How high is the balloon to the nearest foot? 690 ft

www.algebra2.com/self_check_quiz

Reading to Learn Mathematics, p. 797

Reading the Lesson
1. In each case below, the measures of three parts of a triangle are given. For each case, write the formula you would use to find the area of the triangle. Write the formula with the symbols you are given. Then determine what information provided is needed to find the area of the triangle by using the area formula on page 730 in your textbook and without finding other parts of the triangle that aren’t relevant.

- Example: Area of a triangle given \( \angle A = 45^\circ, a = 12, b = 15 \)
- Solution: \( \text{Area} = \frac{1}{2}ab \sin C \)

- Example: Area of a triangle given \( b = 9, c = 12 \)
- Solution: \( \text{Area} = \frac{1}{2}bc \sin A \)

Enrichment, p. 798

Navigation
The bearing of a boat is an angle showing the direction the boat is traveling to its destination. The bearing is measured from the north, and the direction is given in degrees.

Example: A boat sails the lightship in the direction N85°E and the lightship C in the direction S04°E. According to the map, it is 5 miles from the lightship to the lightship.

To find the bearing of the lightship B from the lightship A, follow these steps:

- Step 1: Determine the bearing of the lightship A from the lightship B.
- Step 2: Calculate the distance between the two lightships.
- Step 3: Determine the bearing of the lightship B from the lightship A.

**Helping You Remember**
- Suppose you are traveling a path and cannot determine whether the formula for the area of a triangle is \( \text{Area} = \frac{1}{2}ab \sin C \) or \( \text{Area} = \frac{1}{2}bc \sin A \). Check both. Can you quickly remember which one to use? Sample answer: The formula must work when \( C = 90^\circ \).
Open-Ended Assessment
Modeling Have students work in small groups to build models with coffee stirrer sticks (or similar objects) to illustrate and explain the various number of solutions that are possible for triangles.

Assessment Options
Quiz (Lessons 13-3 and 13-4) is available on p. 831 of the Chapter 13 Resource Masters.
Mid-Chapter Test (Lessons 13-1 through 13-4) is available on p. 833 of the Chapter 13 Resource Masters.

Getting Ready for Lesson 13-5
PREREQUISITE SKILL Lesson 13-5 presents the Law of Cosines. Students will use their familiarity with solving equations involving trigonometric functions as they apply the Law of Cosines. Exercises 55—58 should be used to determine your students’ familiarity with solving equations with trigonometric functions.

Maintain Your Skills

Mixed Review

Find the exact value of each trigonometric function. (Lesson 13-3)

46. \(\cos 30^\circ\) 47. \(\cot \left(\frac{\pi}{3}\right)\)

48. \(\csc \left(\frac{\pi}{4}\right)\sqrt{2}\)

Find one angle with positive measure and one angle with negative measure coterminal with each angle. (Lesson 13-2)

49. 300° 50. 47°

51. \(\frac{5\pi}{6}\), \(-\frac{17\pi}{6}\)

52. Two cards are drawn from a deck of cards. Find each probability. (Lesson 12-5)

53. \(P(\text{both 7s or both red}) = \frac{55}{221}\)

54. AERONAUTICS A rocket rises 20 feet in the first second, 60 feet in the second second, and 100 feet in the third second. If it continues at this rate, how many feet will it rise in the 20th second? (Lesson 11-1) 780 ft

Getting Ready for the Next Lesson

PREREQUISITE SKILL Solve each equation. Round to the nearest tenth. (To review solving equations with trigonometric functions, see Lesson 13-1.)

55. \(a^2 = 3^2 + 5^2 - 2(3)(5)\cos 85^\circ\)

56. \(c^2 = 12^2 + 10^2 - 2(12)(10)\cos 40^\circ\)

57. \(b^2 = 11^2 + 9^2 - 2(11)(9)\cos B^\circ\)

58. \(13^2 = 8^2 + 6^2 - 2(8)(6)\cos A^\circ\)

41. NAVIGATION Two fishing boats, \(A\) and \(B\), are anchored 4500 feet apart in open water. A plane flies at a constant speed in a straight path directly over the two boats, maintaining a constant altitude. At one point during the flight, the angle of depression to \(A\) is 85°, and the angle of depression to \(B\) is 25°. Ten seconds later the plane has passed over \(A\) and spots \(B\) at a 35° angle of depression. How fast is the plane flying? 107 mph

42. CRITICAL THINKING Given \(\triangle ABC\), if \(a = 20\) and \(B = 47^\circ\), then determine all possible values of \(b\) so that the triangle has

- a. two solutions.
- b. one solution.
- c. no solutions.

\[14.63 < b < 20\]  \[b = 14.63\]  \[b \geq 20\]  \[b < 14.63\]

43. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. See margin.

How can trigonometry be used to find the area of a triangle?

Include the following in your answer:

- the conditions that would indicate that trigonometry is needed to find the area of a triangle,
- an example of a real-world situation in which you would need trigonometry to find the area of a triangle, and
- a derivation of one of the other two area formulas.

44. Which of the following is the perimeter of the triangle shown? D

\(\mathbf{A}\) 49.0 cm  \(\mathbf{B}\) 66.0 cm  \(\mathbf{C}\) 91.4 cm  \(\mathbf{D}\) 93.2 cm

45. SHORT RESPONSE The longest side of a triangle is 67 inches. Two angles have measures of 47° and 55°. Solve the triangle. \(B = 78^\circ\), \(a = 50.1\), \(c = 56.1\)

43. Answers should include the following.

- If the height of the triangle is not given, but the measure of two sides and their included angle are given, then the formula for the area of a triangle using the sine function should be used.

- You might use this formula to find the area of a triangular piece of land, since it might be easier to measure two sides and use surveying equipment to measure the included angle than to measure the perpendicular distance from one vertex to its opposite side.

\[
\text{Area} = \frac{1}{2}ah \quad \text{or Area} = \frac{1}{2}a(c \sin B)
\]
5-Minute Check

Transparency 13-5
Use as a quiz or review of Lesson 13-4.

Mathematical Background notes are available for this lesson on p. 698D.

How can you determine the angle at which to install a satellite dish?

Ask students:
• Why does the satellite appear to remain stationary over one point on the equator? because the satellite and Earth are turning at the same rate
• What is the zero point for measuring longitude? The zero meridian of longitude passes through Greenwich, England.

Law of Cosines

Vocabulary
• Law of Cosines

The GE-3 satellite is in a geosynchronous orbit about Earth, meaning that it circles Earth once each day. As a result, the satellite appears to remain stationary over one point on the equator. A receiving dish for the satellite can be directed at one spot in the sky. The satellite orbits 35,786 kilometers above the equator at 87°W longitude. The city of Valparaiso, Indiana, is located at approximately 87°W longitude and 41.5°N latitude.

Knowing the radius of Earth to be about 6375 kilometers, a satellite dish installer can use trigonometry to determine the angle at which to direct the receiver.

LAW OF COSINES Problems such as this, in which you know the measures of two sides and the included angle of a triangle, cannot be solved using the Law of Sines. You can solve problems such as this by using the Law of Cosines.

To derive the Law of Cosines, consider \( \triangle ABC \). What relationship exists between \( a, b, c \), and \( A \)?

\[
\begin{align*}
\text{Use the Pythagorean Theorem for } \triangle DBC. \\
a^2 &= (b - x)^2 + h^2 \\
&= b^2 - 2bx + x^2 + h^2 \\
&= b^2 - 2bx + c^2 \\
&= b^2 - 2b(c \cos A) + c^2 \\
&= b^2 + c^2 - 2bc \cos A \\
\end{align*}
\]

Commutative Property

\[
\begin{align*}
\text{Law of Cosines} \\
a^2 &= b^2 + c^2 - 2bc \cos A \\
b^2 &= a^2 + c^2 - 2ac \cos B \\
c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

Key Concept

Let \( \triangle ABC \) be any triangle with \( a, b, \) and \( c \) representing the measures of sides, and opposite angles with measures \( A, B, \) and \( C \), respectively. Then the following equations are true.

Workbook and Reproducible Masters

Chapter 13 Resource Masters
• Study Guide and Intervention, pp. 799–800
• Skills Practice, p. 801
• Practice, p. 802
• Reading to Learn Mathematics, p. 803
• Enrichment, p. 804

School-to-Career Masters, p. 26

5-Minute Check Transparency 13-5
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 26
Interactive Chalkboard
You can apply the Law of Cosines to a triangle if you know
- the measures of two sides and the included angle, or
- the measures of three sides.

**Example 1** Solve a Triangle Given Two Sides and Included Angle

Solve \( \triangle ABC \).

You are given the measure of two sides and the included angle. Begin by using the Law of Cosines to determine \( c \).

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

Law of Cosines

\[
c^2 = 18^2 + 24^2 - 2(18)(24) \cos 57^
\]

Simplify a calculator.

\[
c^2 = 429.4
\]

\[
c = 20.7
\]

Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle \( A \).

\[
\sin A = \frac{\sin C}{c}
\]

Law of Sines

\[
\sin A = \frac{\sin 57^\circ}{20.7}
\]

\[
a = 18, \quad C = 57^\circ, \quad \text{and} \quad c = 20.7
\]

\[
\sin A = 0.7293
\]

Use a calculator.

\[
A = 47^\circ
\]

Use the \( \sin^{-1} \) function.

The measure of the angle \( B \) is approximately \( 180^\circ - (57^\circ + 47^\circ) \) or \( 76^\circ \).

Therefore, \( c = 20.7, \quad A = 47^\circ, \quad \text{and} \quad B = 76^\circ \).

**Example 2** Solve a Triangle Given Three Sides

Solve \( \triangle ABC \).

You are given the measures of three sides. Use the Law of Cosines to find the measure of the largest angle first, angle \( A \).

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

Law of Cosines

\[
15^2 = 9^2 + 7^2 - 2(9)(7) \cos A
\]

Subtract \( 9^2 \) and \( 7^2 \) from each side.

\[
15^2 - 9^2 - 7^2 = -2(9)(7) \cos A
\]

Divide each side by \(-2(9)(7)\).

\[
-0.7540 = \cos A
\]

Use a calculator.

\[
139^\circ = A
\]

Use the \( \cos^{-1} \) function.

You can use the Law of Sines to find the measure of angle \( B \).

\[
\sin B = \frac{\sin A}{a}
\]

Law of Sines

\[
\sin B = \frac{\sin 139^\circ}{15}
\]

\[
b = 9, \quad A = 139^\circ, \quad \text{and} \quad a = 15
\]

\[
\sin B = 0.9396
\]

Multiply each side by 9.

\[
B = 23^\circ
\]

Use the \( \sin^{-1} \) function.

The measure of the angle \( C \) is approximately \( 180^\circ - (139^\circ + 23^\circ) \) or \( 18^\circ \).

Therefore, \( A = 139^\circ, \quad B = 23^\circ, \quad \text{and} \quad C = 18^\circ \).
EMERGENCY MEDICINE

A medical rescue helicopter has flown from its home base at point C to pick up an accident victim at point B and then from there to the hospital at point A. The pilot needs to know how far he is now from his home base so he can decide whether to refuel before returning. How far is the hospital from the helicopter’s base?

You are given the measures of two sides and their included angle, so use the Law of Cosines to find a.

\[ a^2 = b^2 + c^2 - 2bc \cos A \]

\[ a^2 = 50^2 + 45^2 - 2(50)(45) \cos 130^\circ \]

\[ a^2 = 7417.5 \]

\[ a = 86.1 \]

The distance between the hospital and the helicopter base is approximately 86.1 miles.

www.algebra2.com/extra_examples
2. Explain how to solve a triangle by using the Law of Cosines if the lengths of
a. three sides are known.
b. two sides and the measure of the angle between them are known.

3. OPEN ENDED Give an example of a triangle that can be solved by first using
the Law of Cosines. See margin.

Guided Practice

Determine whether each triangle should be solved by beginning with the
Law of Sines or Law of Cosines. Then solve each triangle. Round measures of
sides to the nearest tenth and measures of angles to the nearest degree.

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4. cosines; $A = 76^\circ$, $B = 69^\circ$, $c = 5.7$

5. sines; $B = 70^\circ$, $a = 9.6$, $b = 14$

6. $A = 42^\circ$, $b = 57$, $a = 63$
sines; $C = 101^\circ$, $B = 37^\circ$, $c = 92.5$

7. $a = 5$, $b = 12$, $c = 13$
cosines; $A = 23^\circ$, $B = 67^\circ$, $C = 90^\circ$

Application

BASEBALL For Exercises 8 and 9, use the following information.
In Australian baseball, the bases lie at the vertices of a square 27.5 meters on a side and the pitcher’s mound is 18 meters from home plate.

8. Find the distance from the pitcher’s mound to first base. 19.5 m

9. Find the angle between home plate, the pitcher’s mound, and first base. 94.3°

Answers

2a. Use the Law of Cosines to find the measure of one angle. Then use the
Law of Sines or the Law of Cosines to find the measure of a second angle. Finally, subtract the sum of
these two angles from 180° to find the measure of the third angle.

2b. Use the Law of Cosines to find the measure of the third side. Then
use the Law of Sines or the Law of
Cosines to find the measure of a
second angle. Finally, subtract the sum of these two angles from
180° to find the measure of the third angle.

3. Sample answer:

About the Exercises...
Organization by Objective
- Law of Cosines: 10–33
- Choose the Method: 10–27

Odd/Even Assignments
Exercises 10–27 are structured
so that students practice the
same concepts whether they
are assigned odd or even
problems.

Assignment Guide
Basic: 11–25 odd, 31, 34–37,
41–54
Average: 11–27 odd, 31, 33–37,
41–54 (optional: 38–40)
Advanced: 10–26 even, 28–30,
32–48 (optional: 49–54)
All: Practice Quiz 2 (1–5)

Answers

2a. Use the Law of Cosines to find the measure of one angle. Then use the
Law of Sines or the Law of Cosines to find the measure of a second angle. Finally, subtract the sum of
these two angles from 180° to find the measure of the third angle.

2b. Use the Law of Cosines to find the measure of the third side. Then
use the Law of Sines or the Law of
Cosines to find the measure of a
second angle. Finally, subtract the sum of these two angles from
180° to find the measure of the third angle.

3. Sample answer:

10. sines; $A = 60^\circ$, $b \approx 14.3$, $c \approx 11.2$

11. cosines; $A \approx 48^\circ$, $B \approx 62^\circ$, $C \approx 70^\circ$

12. cosines; $A \approx 46^\circ$, $B \approx 74^\circ$, $C \approx 59.6$

13. sines; $B = 102^\circ$, $C = 44^\circ$, $b = 21.0$

14. cosines; $A = 56.8^\circ$, $B = 82^\circ$, $c \approx 11.5$

15. sines; $A = 80^\circ$, $a \approx 10.9$, $c = 5.4$

16. cosines; $A = 55^\circ$, $C = 78^\circ$, $b \approx 17.9$

17. cosines; $A \approx 30^\circ$, $B = 110^\circ$, $C = 10^\circ$

18. no

19. sines; $C \approx 77^\circ$, $b \approx 31.7$, $c \approx 31.6$

20. cosines; $A \approx 103^\circ$, $B = 49^\circ$, $C = 28^\circ$

21. no

22. cosines; $A = 15^\circ$, $B \approx 131^\circ$, $C = 34^\circ$

23. cosines; $A \approx 52^\circ$, $C \approx 109^\circ$, $b \approx 21.0$

24. sines; $C = 102^\circ$, $b \approx 5.5$, $c \approx 14.4$

25. cosines; $A \approx 24^\circ$, $B = 125^\circ$, $C = 31^\circ$

26. cosines; $A \approx 107^\circ$, $B \approx 35^\circ$, $c \approx 13.8$

27. cosines; $C = 82^\circ$, $C = 58^\circ$, $a = 4.5$
30. Since the step angle for the carnivore is closer to 180°, it appears as though the carnivore made more forward progress with each step than the herbivore did.

34. Since \( \cos 90° = 0 \), \( a^2 = b^2 + c^2 - 2bc \cos A \) becomes \( a^2 = b^2 + c^2 \).

36. In \( \triangle DEF \), what is the value of \( \theta \) to the nearest degree? 

A) 26°  
B) 74°  
C) 80°  
D) 141°

www.algebra2.com/self_check_quiz

The Law of Cosines and the Pythagorean Theorem

The law of cosines has strong similarities to the Pythagorean Theorem. According to the law of cosines, if two sides of a triangle have lengths \( a \) and \( b \), and the angle between them is \( C \), then the square of the length of the third side can be found by using the equation

\[ c^2 = a^2 + b^2 - 2ab \cos C \]

Answer the following questions to clarify the relationship between the law of cosines and the Pythagorean theorem.

1. If the value of \( C \) becomes less and less, what happens to \( \cos C \) when close to 0°?
2. If the value of \( C \) is very close to 90°, what happens to \( \cos C \) when close to 0°?

Lesson 13-5  Law of Cosines  737
Open-Ended Assessment

Writing Have students sketch and label some of the parts of three triangles—one that they would choose to solve using the Law of Cosines, one that they would choose to solve using the Law of Sines, and one that they would use both laws to solve.

Assessment Options

Practice Quiz 2 The quiz provides students with a brief review of the concepts and skills in Lessons 13-3 through 13-5. Lesson numbers are given to the right of exercises or instruction lines so students can review concepts not yet mastered.

Getting Ready for Lesson 13-6

PREREQUISITE SKILL Lesson 13-6 presents the unit circle approach to trigonometric functions. Students will use their familiarity with coterminal angles as they work with the unit circle. Exercises 49–54 should be used to determine your students’ familiarity with finding coterminal angles.

Answers

40. By finding the measure of angle C in one step using the Law of Cosines, only the given information was used. By finding this angle measure using the Law of Cosines and then the Law of Sines, a calculated value that was not exact was introduced.

43. sin θ = \( \frac{12}{13} \), cos θ = \( \frac{5}{13} \),
   tan θ = \( \frac{12}{5} \), cosec θ = \( \frac{13}{12} \),
   sec θ = \( \frac{13}{5} \), cot θ = \( \frac{5}{12} \).

44. sin θ = \( \frac{\sqrt{6}}{5} \), cos θ = \( \frac{\sqrt{15}}{5} \),
   tan θ = \( \frac{\sqrt{6}}{5} \), cosec θ = \( \frac{\sqrt{15}}{7} \),
   sec θ = \( \frac{5}{4} \), cot θ = \( \frac{4}{\sqrt{6}} \).

45. sin θ = \( \frac{\sqrt{6}}{4} \), cos θ = \( \frac{\sqrt{10}}{4} \),
   tan θ = \( \frac{\sqrt{15}}{5} \), cosec θ = \( \frac{\sqrt{15}}{3} \),
   sec θ = \( \frac{\sqrt{10}}{5} \), cot θ = \( \frac{\sqrt{15}}{3} \).

Practice Quiz 2

1. Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the point (−2, 3). (Lesson 13-3) See margin.
2. Find the exact value of csc \( \frac{5\pi}{3} \). (Lesson 13-3) \( \frac{2\sqrt{3}}{3} \).
3. Find the area of \( \triangle DEF \) to the nearest tenth. (Lesson 13-4) 27.7 m².
4. Determine whether \( \triangle ABC \), with \( A = 22° \), \( a = 15 \), and \( b = 18 \), has no solution, one solution, or two solutions. Then solve the triangle, if possible. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-4) See margin.
5. Determine whether \( \triangle ABC \), with \( b = 11 \), \( c = 14 \), and \( A = 78° \), should be solved by beginning with the Law of Sines or Law of Cosines. Then solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5) cosines; \( C = 53° \), \( B = 43° \).

Answer (Practice Quiz 2)

1. sin θ = \( \frac{3\sqrt{13}}{13} \); cos θ = \( -\frac{2\sqrt{13}}{13} \); tan θ = \( -\frac{3}{2} \);
   csc θ = \( \frac{\sqrt{13}}{3} \); sec θ = \( -\frac{\sqrt{13}}{2} \); cot θ = \( -\frac{2}{3} \).

4. two; \( B = 27° \); \( C \approx 131° \); \( c \approx 30.2 \); \( B \approx 153° \);
   \( C \approx 5° \); \( c \approx 3.5 \).
5-Minute Check

Use as a quiz or review of Lesson 13-5.

Mathematical Background notes are available for this lesson on p. 698D.

How can you model annual temperature fluctuations?

Ask students:
• Describe the change in the average high temperature from February to July. It is increasing.
• Describe the change in the average high temperature from August to December. It is decreasing.
• Does the pattern of change in temperature seem to fall along a straight line? No
• What curve does the pattern of change in temperature seem to suggest? a sine or cosine curve

UNIT CIRCLE DEFINITIONS From your work with reference angles, you know that the values of trigonometric functions also repeat. For example, sin 30° and sin 150° have the same value, \(\frac{1}{2}\). In this lesson, we will further generalize the trigonometric functions by defining them in terms of the unit circle.

Consider an angle \(\theta\) in standard position. The terminal side of the angle intersects the unit circle at a unique point, \(P(x, y)\). Recall that \(\sin \theta = \frac{y}{r}\) and \(\cos \theta = \frac{x}{r}\). Since \(P(x, y)\) is on the unit circle, \(r = 1\). Therefore, \(\sin \theta = y\) and \(\cos \theta = x\).

Key Concept

**Definition of Sine and Cosine**

If the terminal side of an angle \(\theta\) in standard position intersects the unit circle at \(P(x, y)\), then \(\cos \theta = x\) and \(\sin \theta = y\). Therefore, the coordinates of \(P\) can be written as \(P(\cos \theta, \sin \theta)\).
UNIT CIRCLE DEFINITIONS

In-Class Example

Example 1: Find Sine and Cosine Given Point on Unit Circle

Given an angle \( \theta \) in standard position, if \( P \left( \frac{\sqrt{2}}{3}, -\frac{1}{3} \right) \) lies on the terminal side and on the unit circle, find \( \sin \theta \) and \( \cos \theta \).

\( P \left( \frac{2\sqrt{2}}{3}, -\frac{1}{3} \right) = P(\cos \theta, \sin \theta) \),

so \( \sin \theta = -\frac{1}{3} \) and \( \cos \theta = \frac{2\sqrt{2}}{3} \).

In the Investigation below, you will explore the behavior of the sine and cosine functions on the unit circle.

Graphing Calculator Investigation

Sine and Cosine on the Unit Circle

Press MODE on a TI-83 Plus and highlight Degree and Par. Then use the following range values to set up a viewing window: TMIN = 0, TMAX = 360, TSTEP = 15, XMIN = -2.4, XMAX = 2.55, XSCL = 0.5, YMIN = -1.5, YMAX = 1.55, YSCL = 0.5.

Press Y= to define the unit circle with \( X_T = \cos \theta \) and \( Y_T = \sin \theta \). Press GRAPH. Use the TRACE function to move around the circle.

Think and Discuss

1. What does \( T \) represent? What does the \( x \) value represent? What does the \( y \) value represent? the angle \( \theta \); \( \cos \theta \); \( \sin \theta \)
2. Determine the sine and cosine of the angles whose terminal sides lie at \( 0^\circ \), \( 90^\circ \), \( 180^\circ \), and \( 270^\circ \).
3. How do the values of sine change as you move around the unit circle? How do the values of cosine change?

The exact values of the sine and cosine functions for specific angles are summarized using the definition of sine and cosine on the unit circle below.

Graphing Calculator Investigation

Sine and Cosine on the Unit Circle Follow the given steps for the TI-83 calculator also. Point out that, after the mode and window have been defined, from the Y= screen, pressing the \[ X \cdot T \cdot n \] key will automatically enter T. After pressing TRACE, use \[ \uparrow \] to move around the circle counterclockwise. Call attention to the information that appears on the screen as the cursor moves around the circle.
PERIODIC FUNCTIONS

Notice in the graph above that the values of sine for the coterminal angles $0^\circ$ and $360^\circ$ are both 0. The values of cosine for these angles are both 1. Every $360^\circ$ or $2\pi$ radians, the sine and cosine functions repeat their values. So, we can say that the sine and cosine functions are periodic, each having a period of $360^\circ$ or $2\pi$ radians.

For the sine and cosine functions, $\cos (x + 360^\circ) = \cos x$, and $\sin (x + 360^\circ) = \sin x$. In radian measure, $\cos (x + 2\pi) = \cos x$, and $\sin (x + 2\pi) = \sin x$. Therefore, the period of the sine and cosine functions is $360^\circ$ or $2\pi$.

Example 2 Find the Value of a Trigonometric Function

Find the exact value of each function.

a. $\cos 675^\circ$

$\cos 675^\circ = \cos (315^\circ + 360^\circ)$

$= \cos 315^\circ$

$= \frac{\sqrt{2}}{2}$

b. $\sin \left(-\frac{5\pi}{6}\right)$

$\sin \left(-\frac{5\pi}{6}\right) = \sin \left(-\frac{5\pi}{6} + 2\pi\right)$

$= \sin \left(-\frac{5\pi}{6} + \frac{12\pi}{6}\right)$

$= \sin \left(-\frac{5\pi}{6} + \frac{12\pi}{6}\right)$

$= \sin \frac{7\pi}{6}$

$= \frac{1}{2}$

When you look at the graph of a periodic function, you will see a repeating pattern: a shape that repeats over and over as you move to the right on the x-axis. The period is the distance along the x-axis from the beginning of the pattern to the point at which it begins again.

www.algebra2.com/extra_examples
In-Class Example

3 FERRIS WHEEL On another Ferris wheel, the diameter is 42 feet, and it travels at a rate of 3 revolutions per minute.

a. Identify the period of this function. **20 seconds**
b. Make a graph in which the horizontal axis represents the time \( t \) in seconds and the vertical axis represents the height \( h \) in feet in relation to the starting point.

---

**Answer**

1. The terminal side of the angle \( \theta \) in standard position must intersect the unit circle at \( P(x, y) \).

---

**Check for Understanding**

**Concept Check**

2. Sample answer: the motion of the minute hand on a clock; 60 s

**Guided Practice**

<table>
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<td>8–10</td>
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</table>

1. State the conditions under which \( \cos \theta = x \) and \( \sin \theta = y \). See margin.
2. **OPEN ENDED** Give an example of a situation that could be described by a periodic function. Then state the period of the function.
3. **Compare and contrast** the graphs of the sine and cosine functions on page 741. Sample answer: The graphs have the same shape, but cross the \( x \)-axis at different points.
4. If the given point \( P \) is located on the unit circle, find \( \sin \theta \) and \( \cos \theta \).
   - \( P \left( \frac{5}{13}, -\frac{12}{13} \right) \)  \( \sin \theta = -\frac{12}{13} \); \( \cos \theta = \frac{5}{13} \)
   - \( P \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \)  \( \sin \theta = -\frac{\sqrt{2}}{2} \); \( \cos \theta = \frac{\sqrt{2}}{2} \)

Find the exact value of each function.

5. \( \sin \left( -240^\circ \right) = \frac{\sqrt{3}}{2} \)
6. \( \cos \left( \frac{10\pi}{3} \right) = -\frac{1}{2} \)

---

**Example 3** Find the Value of a Trigonometric Function

**FERRIS WHEEL** As you ride a Ferris wheel, the height that you are above the ground varies periodically as a function of time. Consider the height of the center of the wheel to be the starting point. A particular wheel has a diameter of 38 feet and travels at a rate of 4 revolutions per minute.

a. Identify the period of this function.
   - Since the wheel makes 4 complete counterclockwise rotations every minute, the period is the time it takes to complete one rotation, which is \( \frac{1}{4} \) of a minute or 15 seconds.

b. Make a graph in which the horizontal axis represents the time \( t \) in seconds and the vertical axis represents the height \( h \) in feet in relation to the starting point.
   - Since the diameter of the wheel is 38 feet, the wheel reaches a maximum height of \( \frac{38}{2} \) or 19 feet above the starting point and a minimum of 19 feet below the starting point.

---

**Differentiated Instruction**

**Naturalist** Have students research various kinds of circular calendars, such as those used by the Mayans, to predict the weather and determine the best time for planting crops.
Application  
**PHYSICS**  
For Exercises 9 and 10, use the following information.  
The motion of a weight on a spring varies periodically as a function of time. Suppose you pull the weight down 3 inches from its equilibrium point and then release it. It bounces above the equilibrium point and then returns below the equilibrium point in 2 seconds.  
9. Find the period of this function.  
10. Graph the height of the spring as a function of time.  
See margin.

**Practice and Apply**

The given point \( P \) is located on the unit circle. Find \( \sin \theta \) and \( \cos \theta \).

- 11. \( P \left( \frac{3}{5}, \frac{4}{5} \right) \)
- 12. \( P \left( \frac{12}{13}, -\frac{5}{13} \right) \)
- 13. \( P \left( \frac{8}{17}, \frac{15}{17} \right) \)
- 14. \( P \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \)
- 15. \( P \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \)
- 16. \( P(0.6, 0.8) \)

Find the exact value of each function.

- 17. \( \sin 60^\circ + 1\frac{\sqrt{3}}{2} \)
- 18. \( \cos 750^\circ \)
- 19. \( \cos 5\pi \)
- 20. \( \sin \left( \frac{14\pi}{6} \right) \)
- 21. \( \sin \left( -\frac{3\pi}{2} \right) \)
- 22. \( \cos (-225^\circ) \)

Determine the period of each function.

- 23. \( \cos 60^\circ + \sin 30^\circ \)
- 24. \( 3\sin 60^\circ(\cos 30^\circ) \)
- 25. \( \sin 30^\circ - \sin 60^\circ \)
- 26. \( 4\cos 330^\circ + 2\sin 60^\circ \)
- 27. \( 12(\sin 150^\circ)(\cos 150^\circ) \)
- 28. \( (\sin 30^\circ)^2 + (\cos 30^\circ)^2 \)

**Extra Practice**  
See page 858.  
11. \( \sin \theta = \frac{4}{5}; \cos \theta = -\frac{3}{5} \)
12. \( \sin \theta = -\frac{5}{13}; \cos \theta = -\frac{12}{13} \)
13. \( \sin \theta = \frac{15}{17}; \cos \theta = \frac{8}{17} \)
14. \( \sin \theta = -\frac{1}{2}; \cos \theta = \frac{\sqrt{3}}{2} \)
15. \( \sin \theta = \frac{\sqrt{3}}{2}; \cos \theta = -\frac{1}{2} \)
16. \( \sin \theta = 0.8; \cos \theta = 0.6 \)

**Homework Help**  
For Exercises  |  See Examples  
--- | ---  
11–16 | 1  
17–28 | 2  
29–42 | 3

**Extra Practice**  
See page 858.  
11. \( \sin \theta = \frac{4}{5}; \cos \theta = -\frac{3}{5} \)
12. \( \sin \theta = -\frac{5}{13}; \cos \theta = -\frac{12}{13} \)
13. \( \sin \theta = \frac{15}{17}; \cos \theta = \frac{8}{17} \)
14. \( \sin \theta = -\frac{1}{2}; \cos \theta = \frac{\sqrt{3}}{2} \)
15. \( \sin \theta = \frac{\sqrt{3}}{2}; \cos \theta = -\frac{1}{2} \)
16. \( \sin \theta = 0.8; \cos \theta = 0.6 \)

**Study Notebook**  
Have students—  
- add the definitions/examples of the vocabulary terms to their  
  Vocabulary Builder worksheets for  
  Chapter 13.  
- include any other item(s) that they find helpful in mastering the skills in this lesson.

**About the Exercises...**  
**Organization by Objective**  
- Unit Circle Definitions: 11–16  
- Periodic Functions: 17–42  

**Odd/Even Assignments**  
Exercises 11–32 are structured so that students practice the same concepts whether they are assigned odd or even problems.

**Assignment Guide**  
**Basic:** 11–21 odd, 29, 31, 33, 34, 43–69  
**Average:** 11–31 odd, 33–35, 43–69  
**Advanced:** 12–32 even, 33, 34, 36–63 (optional: 64–69)
Determine the period of each function.

\[
\cos \theta
\]

Find the exact value of each function.

1. If the terminal side and on the unit circle, find sin \(\theta\) and cos \(\theta\).
2. If \(P\) is on the terminal side of \(\theta = 7\), find the exact coordinates of the remaining vertices.
3. If \(P\) is on the terminal side of \(\theta = 30\), find the exact coordinates of the remaining vertices.
4. If \(P\) is on the terminal side of \(\theta = 210\), find the exact coordinates of the remaining vertices.

Most guitars have six strings. The frequency at which one of these strings vibrates is controlled by the length of the string, the amount of tension on the string, the weight of the string, and springiness of the strings’ material.

Source: www.howstuffworks.com

**Reading to Learn Mathematics, p. 809**

**ELL**

**Pre-Activity:** How can you model annual temperature fluctuations?

Read the introduction to Lesson 13.4 at the top of page 780 in your textbook.

- If the graph in your textbook is used, how exactly will it reflect the
  - change? What is the range of this change?
- What is the range of the changes?

**Reading the Lesson:**

1. Use the unit circle on page 780 in your textbook to find the exact values of each expression:
   - \(\cos \theta\)
   - \(\sin \theta\)
   - \(\tan \theta\)
   - \(\cot \theta\)
   - \(\sec \theta\)
   - \(\csc \theta\)

2. Determine the period of each function.

3. Ferris Wheel: A Ferris wheel has a diameter of 500 feet complete 2.5 revolutions per minute. How is the period of the function that describes the height of a seat on the outside of the Ferris Wheel in terms of time? 20 s

**Enrichment, p. 810**

**Polar Coordinates:** Consider an angle in standard position with its vertex at a point \(O\) and its initial side on a coordinate axis called the polar axis. A point \(P\) is then located in the plane so that \(OP = r\). The Cartesian coordinates \((x, y)\) of point \(P\) are related to the polar coordinates \((r, \theta)\) by the equations:

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

**Polar Form:** Any complex number of the form \(a + bi\) can be represented by a point \((r, \theta)\) in the complex plane. The polar form of a complex number \(a + bi\) is:

\[
r \exp(\theta i) = r \cos \theta + r \sin \theta
\]

**Answer:**

36. The population is around 425 near the 60th day of the year. It rises to around 625 in May/June. It falls to around 425 again by August/September. It continues to drop to around 225 in November/December.

**Critical Thinking:** Determine the domain and range of the functions:

- \(y = \sin \theta\)
- \(y = \cos \theta\)

Include the following in your answer:
- a description of how the sine and cosine functions are similar to annual temperature fluctuations, and
- if the formula for the temperature \(T\) in degrees Fahrenheit of a city \(t\) months into the year is given by \(T = 50 + 25 \sin \left(\frac{\pi t}{6}\right)\), explain how to find the average temperature and the maximum and minimum predicted over the year.

**Writing in Math**

Answer the question that was posed at the beginning of the lesson. See margin.

How can you model annual temperature fluctuations?

Include the following in your answer:
- a description of how the sine and cosine functions are similar to annual temperature fluctuations, and
- if the formula for the temperature \(T\) in degrees Fahrenheit of a city \(t\) months into the year is given by \(T = 50 + 25 \sin \left(\frac{\pi t}{6}\right)\), explain how to find the average temperature and the maximum and minimum predicted over the year.

**Skills Practice, p. 808 and Practice, p. 808 (shown)**

The line \(x = 2\) is given.

Find the exact value of each function:

\[
\begin{align*}
\sin \theta &= \frac{1}{2} \\
\cos \theta &= \frac{\sqrt{3}}{2}
\end{align*}
\]

1. If a point \((x, y)\) is on the terminal side of \(\theta\) and \(x = 2\), find the coordinates of the remaining vertices.
2. If \(\theta = 30\), find the coordinates of the remaining vertices.
3. If \(\theta = 210\), find the coordinates of the remaining vertices.
4. If \(\theta = 210\), find the coordinates of the remaining vertices.

**Slope**

For Exercises 37–42, use the following information.

Suppose the terminal side of an angle \(\theta\) in standard position intersects the unit circle at \(P(x, y)\).

37. What is the slope of \(OP\)?

38. Which of the six trigonometric functions is equal to the slope of \(OP\)?

39. What is the slope of any line perpendicular to \(OP\)?

40. Which of the six trigonometric functions is equal to the slope of any line perpendicular to \(OP\)?

41. Find the slope of \(OP\) when \(\theta = 60\).

42. If \(\theta = 60\), find the slope of the line tangent to circle \(O\) at point \(P\).

**Critical Thinking**

Determine the domain and range of the functions:

\[
\begin{align*}
\cos \theta &= {all reals}, R \\{0, 1\} & \sin \theta &= {all reals}, R \\{0, 1\}
\end{align*}
\]

44. **Writing in Math**

Answer the question that was posed at the beginning of the lesson. See margin.

How can you model annual temperature fluctuations?

Include the following in your answer:

- a description of how the sine and cosine functions are similar to annual temperature fluctuations, and
- if the formula for the temperature \(T\) in degrees Fahrenheit of a city \(t\) months into the year is given by \(T = 50 + 25 \sin \left(\frac{\pi t}{6}\right)\), explain how to find the average temperature and the maximum and minimum predicted over the year.

**Study Guide and Intervention**

**Chapter 13 Trigonometric Functions**

**Enlarged Picture:** Draw the graph of the function.

**Table of Corresponding Values:**

\[
\begin{array}{|c|c|c|c|c|}
\hline
\theta & \cos \theta & \sin \theta & \tan \theta & \cot \theta \\
\hline
\theta_1 & -1 & 0 & 0 & -1 \\
\theta_2 & 0 & 1 & 0 & 0 \\
\theta_3 & 1 & 0 & \infty & 0 \\
\hline
\end{array}
\]

4. **Critical Thinking**

Determine the domain and range of the functions:

\[
\begin{align*}
\cos \theta &= {all reals}, R \\{0, 1\} & \sin \theta &= {all reals}, R \\{0, 1\}
\end{align*}
\]

44. **Writing in Math**

Answer the question that was posed at the beginning of the lesson. See margin.

How can you model annual temperature fluctuations?

Include the following in your answer:

- a description of how the sine and cosine functions are similar to annual temperature fluctuations, and
- if the formula for the temperature \(T\) in degrees Fahrenheit of a city \(t\) months into the year is given by \(T = 50 + 25 \sin \left(\frac{\pi t}{6}\right)\), explain how to find the average temperature and the maximum and minimum predicted over the year.

**4. Critical Thinking**

Determine the domain and range of the functions:

\[
\begin{align*}
\cos \theta &= {all reals}, R \\{0, 1\} & \sin \theta &= {all reals}, R \\{0, 1\}
\end{align*}
\]

44. **Writing in Math**

Answer the question that was posed at the beginning of the lesson. See margin.

How can you model annual temperature fluctuations?

Include the following in your answer:

- a description of how the sine and cosine functions are similar to annual temperature fluctuations, and
- if the formula for the temperature \(T\) in degrees Fahrenheit of a city \(t\) months into the year is given by \(T = 50 + 25 \sin \left(\frac{\pi t}{6}\right)\), explain how to find the average temperature and the maximum and minimum predicted over the year.
45. If \( \triangle ABC \) is an equilateral triangle, what is the length of \( AD \), in units?  
   \[ \text{A} \quad 5 \sqrt{2} \]
   \[ \text{B} \quad 5 \]
   \[ \text{C} \quad 10 \sqrt{2} \]
   \[ \text{D} \quad 10 \]

46. **SHORT RESPONSE** What is the exact value of \( \tan 1830^\circ \)? \( \frac{\sqrt{3}}{3} \)

---

### Maintain Your Skills

#### Mixed Review

**BULBS** For Exercises 51–56, use the following information. The lifetimes of 10,000 light bulbs are normally distributed. The mean lifetime is 300 days, and the standard deviation is 40 days. *(Lesson 12-7)*

51. How many light bulbs will last 260 and 340 days? 6800
52. How many light bulbs will last between 220 and 380 days? 9500
53. How many light bulbs will last fewer than 300 days? 5000
54. How many light bulbs will last more than 300 days? 5000
55. How many light bulbs will last more than 380 days? 250
56. How many light bulbs will last fewer than 180 days? 50

Find the area of \( \triangle ABC \). Round to the nearest tenth. *(Lesson 13-4)*

47. \( a = 11 \) in., \( b = 5 \) in., \( c = 7 \) in., \( A = 63^\circ \) \( 12.5 \) in²

48. \( a = 12.4 \), \( B = 59^\circ \), \( A = 76^\circ \) \( \cos \) \( c \approx 12 \)

Find the sum of each infinite geometric series, if it exists. *(Lesson 11-5)*

57. \( a_1 = 3 \), \( r = 1.2 \) does not exist
58. \( 16 \), \( 4 \), \( 1 \), \( \frac{1}{4} \), … \( \frac{64}{3} \)
59. \( \sum_{n=1}^{\infty} 13(-0.625)^n - 1 \) \( 8 \)

Use synthetic division to find each quotient. *(Lesson 5-3)*

60. \( (4x^2 - 13x + 10) \div (x - 2) \) \( 4x - 5 \)
61. \( (2x^2 + 21x + 54) \div (x + 6) \) \( 2x + 9 \)
62. \( (5y^3 + y^2 - 7) \div (y + 1) \) \( 5y^2 - 4y + 4 - \frac{11}{y + 1} \)
63. \( (2y^2 + y - 16) \div (y - 3) \) \( 2y + 7 + \frac{5}{y - 3} \)

**PREPARING THE NEXT LESSON** Find each value of \( \theta \). Round to the nearest degree. *(Lesson 13-1)*

64. \( \sin \theta = 0.3420 \) \( 20^\circ \)
65. \( \cos \theta = -0.3420 \) \( 110^\circ \)
66. \( \tan \theta = 3.2709 \) \( 73^\circ \)
67. \( \tan \theta = 5.6713 \) \( 80^\circ \)
68. \( \sin \theta = 0.8290 \) \( 56^\circ \)
69. \( \cos \theta = 0.0175 \) \( 89^\circ \)

---

**Open-Ended Assessment**

**Modeling** Have students use a geoboard to make a model of a regular octagon inscribed in a unit circle, similar to that shown in Exercise 35, and find the exact coordinates of the vertices.

**Assessment Options**

**Quiz (Lessons 13-5 and 13-6)** is available on p. 832 of the Chapter 13 Resource Masters.

---

**Getting Ready for Lesson 13-7**

**PREPARING THE NEXT LESSON** Find each value of \( \theta \). Round to the nearest degree. *(Lesson 13-1)*

44. Answers should include the following.
   - Over the course of one period both the sine and cosine function attain their maximum value once and their minimum value once. From the maximum to the minimum the functions decrease slowly at first, then decrease more quickly and return to a slow rate of change as they come into the minimum. Similarly, the functions rise slowly from their minimum. They begin to increase more rapidly as they pass the halfway point, and then begin to rise more slowly as they increase into the maximum. Annual temperature fluctuations behave in exactly the same manner.
   - The maximum value of the sine function is 1 so the maximum temperature would be 50 + 25(1) or 75°F. Similarly, the minimum value would be 50 + 25(-1) or 25°F. The average temperature over this time period occurs when the sine function takes on a value of 0. In this case that would be 50°F.
What You’ll Learn

• Solve equations by using inverse trigonometric functions.

• Find values of expressions involving trigonometric functions.

How are inverse trigonometric functions used in road design?

When a car travels a curve on a horizontal road, the friction between the tires and the road keeps the car on the road. Above a certain speed, however, the force of friction will not be great enough to hold the car in the curve. For this reason, civil engineers design banked curves.

The proper banking angle \( \theta \) for a car making a turn of radius \( r \) feet at a velocity \( v \) in feet per second is given by the equation

\[
\tan \theta = \frac{v^2}{2r}.
\]

In order to determine the appropriate value of \( \theta \) for a specific curve, you need to know the radius of the curve, the maximum allowable velocity of cars making the curve, and how to determine the angle \( \theta \) given the value of its tangent.

SOLVE EQUATIONS USING INVERSES

Sometimes the value of a trigonometric function for an angle is known and it is necessary to find the measure of the angle. The concept of inverse functions can be applied to find the inverse of trigonometric functions.

In Lesson 8-8, you learned that the inverse of a function is the relation in which all the values of \( x \) and \( y \) are reversed. The graphs of \( y = \sin x \) and its inverse, \( x = \sin y \), are shown below.

Notice that the inverse is not a function, since it fails the vertical line test. None of the inverses of the trigonometric functions are functions.

We must restrict the domain of trigonometric functions so that their inverses are functions. The values in these restricted domains are called principal values. Capital letters are used to distinguish trigonometric functions with restricted domains from the usual trigonometric functions.

Vocabulary

• principal values
• Arcsine function
• Arccosine function
• Arctangent function

Ask students:

• Where else have you seen banking on curves used?
  Sample answers: skating, skateboarding, and other sports

• The force of gravity is 32 feet per second per second. Where does gravity appear in the formula? in the denominator
The inverse of the Sine function is called the **Arcsine function** and is symbolized by $\text{Sin}^{-1}$ or $\text{Arcsin}$. The Arcsine function has the following characteristics.

- Its domain is the set of real numbers from $-1$ to $1$.
- Its range is the set of angle measures from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
- $\text{Sin} x = y$ if and only if $\text{Sin}^{-1} y = x$.
- $[\text{Sin}^{-1} \circ \text{Sin}](x) = [\text{Sin} \circ \text{Sin}^{-1}](x) = x$.

The definitions of the Arcosine and Arctangent functions are similar to the definition of the Arcsine function.

### Example 1 Solve an Equation

Solve $\text{Sin} x = \frac{\sqrt{2}}{2}$ by finding the value of $x$ to the nearest degree.

If $\text{Sin} x = \frac{\sqrt{2}}{2}$, then $x$ is the least value whose sine is $\frac{\sqrt{2}}{2}$. So, $x = \text{Arcsin} \frac{\sqrt{2}}{2}$.

Use a calculator to find $x$.

**KEYSTROKES:**

\[
\text{2nd} \ [\text{SIN}^{-1}] \ [\text{2nd}] \ [\sqrt{\text{3}}] \ \div \ [\text{2}] \ \text{ENTER} \ 60
\]

Therefore, $x = 60^\circ$.

[www.algebra2.com/extra_examples](http://www.algebra2.com/extra_examples)
Many application problems involve finding the inverse of a trigonometric function.

Example 2 Apply an Inverse to Solve a Problem

DRAWBRIDGE Each leaf of a certain double-leaf drawbridge is 130 feet long. If an 80-foot wide ship needs to pass through the bridge, what is the minimum angle \( \theta \), to the nearest degree, which each leaf of the bridge should open so that the ship will fit?

When the two parts of the bridge are in their lowered position, the bridge spans 130 or 260 feet. In order for the ship to fit, the distance between the leaves must be at least 80 feet.

This leaves a horizontal distance of \( \frac{260 - 80}{2} \) or 90 feet from the pivot point of each leaf to the ship as shown in the diagram at the right.

To find the measure of angle \( \theta \), use the cosine ratio for right triangles.

\[
\cos \theta = \frac{\text{adj}}{\text{hyp}}
\]

Replace adj with 90 and hyp with 130.

\[
\theta = \cos^{-1} \left( \frac{90}{130} \right)
\]

Use a calculator.

Thus, the minimum angle through which each leaf of the bridge should open is 47°.

Example 3 Find a Trigonometric Value

Find each value. Write angle measures in radians. Round to the nearest hundredth.

a. Arcsin \( \frac{\sqrt{2}}{2} \) \( \approx \) 0.79 radians

b. \( \tan \left( \cos^{-1} \frac{4}{5} \right) \) 0.75 radians

TRIGONOMETRIC VALUES You can use a calculator to find the values of trigonometric expressions.

Study Tip Angle Measure

Remember that when evaluating an inverse trigonometric function the result is an angle measure.

Keystrokes: [2nd] [SIN] [INV] 3 [2nd] [√] 2 [ENTER] 1.047197551

Therefore, Arcsin \( \frac{\sqrt{3}}{2} \) \( = \) 1.05 radians.

b. \( \tan \left( \cos^{-1} \frac{6}{7} \right) \)

Keystrokes: [TAN] [2nd] [COS] [INV] 6 [÷] 7 [ENTER] 0.6009252126

Therefore, tan \( \left( \cos^{-1} \frac{6}{7} \right) \) \( = \) 0.60.

Differentiated Instruction

Visual/Spatial Ask students to find Arcsin 2. If they use a calculator, suggest that they study the graph of \( y = \sin x \) to explain why an error message was the result. The graph of \( y = \sin x \) has no y values greater than 1 or less than -1.
Concept Check
1. Explain how you know when the domain of a trigonometric function is restricted. **Restricted domains are denoted with a capital letter.**

2. **OPEN ENDED** Write an equation giving the value of the Cosine function for an angle measure in its domain. Then, write your equation in the form of an inverse function.

3. Describe how $y = \cos x$ and $y = \arccos x$ are related. **They are inverses of each other.**

Guided Practice

Write each equation in the form of an inverse function.

4. $\tan \theta = x \quad \theta = \arctan x$

5. $\cos \alpha = 0.5 \quad \alpha = \arccos 0.5$

Solve each equation by finding the value of $x$ to the nearest degree.

6. $x = \cos^{-1} \frac{\sqrt{2}}{2} \quad 45^\circ$

7. $\arctan x = 0^\circ$

Find each value. Write angle measures in radians. Round to the nearest hundredth.

8. $\tan^{-1} \left( -\frac{\sqrt{3}}{3} \right) \quad 9. \cos^{-1}(-1) = \pi \quad 10. \cos \left( \cos^{-1} \frac{3}{9} \right) = 0.22$

11. $\sin^{-1} \frac{3}{4} = 0.75 \quad 12. \sin \left( \cos^{-1} \frac{3}{4} \right) = 0.66 \quad 13. \tan \left( \sin^{-1} \frac{1}{2} \right) = 0.58$

Application

14. ARCHITECTURE The support for a roof is shaped like two right triangles as shown at the right. Find $\theta$. $30^\circ$

Practice and Apply

Write each equation in the form of an inverse function. 15–20. See margin.

15. $\alpha = \sin \beta$

16. $\tan a = b$

17. $\cos y = x$

18. $\sin 30^\circ = \frac{1}{2}$

19. $\cos 45^\circ = y$

20. $-\frac{4}{3} = \tan x$

Solve each equation by finding the value of $x$ to the nearest degree.

21. $x = \cos^{-1} \frac{1}{2} = 60^\circ$

22. $\sin^{-1} \frac{1}{2} = x \quad 30^\circ$

23. $\arctan 1 = x \quad 45^\circ$

24. $x = \arctan \frac{\sqrt{3}}{3} \quad 30^\circ$

25. $x = \sin^{-1} \frac{1}{2} \quad 45^\circ$

26. $x = \cos^{-1} 0 \quad 90^\circ$

Find each value. Write angle measures in radians. Round to the nearest hundredth.

27. $\cos^{-1} \left( -\frac{1}{2} \right) = 2.09$

28. $\sin^{-1} \frac{\pi}{2} = 29. \arctan \frac{\sqrt{3}}{3} = 0.52$

30. $\arccos \frac{\sqrt{3}}{2} = 0.52$

31. $\sin \left( \sin^{-1} \frac{1}{2} \right) = 0.5$

32. $\cot \left( \sin^{-1} \frac{5}{6} \right) = 0.66$

33. $\tan \left( \cos^{-1} \frac{6}{7} \right) = 0.60$

34. $\sin \left( \arctan \frac{\sqrt{3}}{3} \right) = 0.5$

35. $\cos \left( \arcsin \frac{3}{5} \right) = 0.8$

36. $\cot \left( \sin^{-1} \frac{7}{9} \right) = 0.81$

37. $\cos \left( \tan^{-1} \sqrt{3} \right) = 0.5$

38. $\tan \left( \arctan 3 \right) = 3$

41. $0.71$

42. $\cos^{-1} \left( \sin^{-1} \frac{1}{2} \right) = 0.5$

43. $\sin \left( 2 \cos^{-1} \frac{3}{5} \right) = 0.96$

44. $\sin \left( 2 \sin^{-1} \frac{1}{2} \right) = 0.87$

42. does not exist

www.algebra2.com/self_check_quiz

Lesson 13-7 Inverse Trigonometric Functions

Answers

15. $\beta = \arcsin \alpha$

16. $a = \arctan b$

17. $y = \arccos x$

18. $30^\circ = \arccos \frac{1}{2}$

19. $\arccos y = 45^\circ$

20. $\arctan \left( -\frac{4}{3} \right) = x$
47. No; with this point on the terminal side of the throwing angle \( \theta \), the measure of \( \theta \) is found by solving the equation \( \tan \theta = 17 \). Thus \( \theta = \tan^{-1} 17 \) or about 43.3\(^\circ\), which is greater than the 40\(^\circ\) requirement.

46. **Fountains**
Architects who design fountains know that both the height and distance that a water jet will project is dependent on the angle \( \theta \) at which the water is aimed. For a given angle \( \theta \), the ratio of the maximum height \( H \) of the parabolic arc to the horizontal distance \( D \) it travels is given by \( \frac{H}{D} = \frac{1}{4} \tan \theta \). Find the value of \( \theta \), to the nearest degree, that will cause the arc to go twice as high as it travels horizontally. 83°

47. **TRACK AND FIELD**
When a shot put is thrown, it must land in a 40\(^\circ\) sector. Consider a coordinate system in which the vertex of the sector is at the origin and one side lies along the x-axis. If an athlete puts the shot so that it lands at a point with coordinates (18, 17), did the shot land in the required region? Explain your reasoning.

48. **OPTICS**
You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose horizontally-polarized light with intensity \( I_0 \) strikes a polarizer filter with its axis at an angle \( \theta \) with the horizontal. The intensity of the transmitted light \( I_t \) and \( \theta \) are related by the equation \( \cos \theta = \sqrt{\frac{I_t}{I_0}} \). If one fourth of the polarized light is transmitted through the lens, what does the transmission axis of the filter make with the horizontal? 60°

**CRITICAL THINKING**
For Exercises 49–51, use the following information.
If the graph of the line \( y = mx + b \) intersects the \( x \)-axis such that an angle of \( \theta \) is formed with the positive \( x \)-axis, then \( \tan \theta = m \).

49. Find the acute angle that the graph of \( 3x + 5y = 7 \) makes with the positive \( x \)-axis. 31°

50. Determine the obtuse angle formed at the intersection of the graphs of \( 2x + 5y = 8 \) and \( 6x - y = -8 \). State the measure of the angle to the nearest degree. 102°

51. Explain why this relationship, \( \tan \theta = m \), holds true. See margin.
52. **Writing in Math** Answer the question that was posed at the beginning of the lesson. See margin.

How are inverse trigonometric functions used in road design?

Include the following in your answer:
- A few sentences describing how to determine the banking angle for a road, and
- A description of what would have to be done to a road if the speed limit were increased and the banking angle was not changed.

**Standardized Test Practice**

53. **GRID IN** Find the angle of depression \( \theta \) between the shallow end and the deep end of the swimming pool to the nearest degree. 37°

[Diagram of the swimming pool with dimensions and angles labeled]

54. If \( \sin \theta = \frac{2}{3} \) and \(-90° \leq \theta \leq 90°\), then \( \cos 2\theta = ? \)

\( \text{A} \), \(-\frac{1}{9}\); \( \text{B} \), \(-\frac{1}{3}\); \( \text{C} \), \(\frac{1}{3}\); \( \text{D} \), \(\frac{1}{9}\); \( \text{E} \), 1.

**Graphing Calculator**

55. Copy and complete the table below by evaluating \( y \) for each value of \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1/2</th>
<th>( \sqrt{2}/2 )</th>
<th>( \sqrt{3}/2 )</th>
<th>1</th>
<th>(-1)</th>
<th>(-\sqrt{2}/2)</th>
<th>(-\sqrt{3}/2)</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( \pi )</td>
<td>2</td>
<td>( \pi )</td>
<td>2</td>
<td>( \pi )</td>
<td>2</td>
<td>( \pi )</td>
<td>2</td>
<td>( \pi )</td>
</tr>
</tbody>
</table>

56. \( \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \) for all values of \( x \).

56. Make a conjecture about the function \( y = \sin^{-1} x + \cos^{-1} x \).

57. Considering only positive values of \( x \), provide an explanation of why your conjecture might be true. See margin.

**Maintain Your Skills**

**Mixed Review** (Lesson 13-6)

58. \( \sin -660° \sqrt{3}/2 \) 59. \( \cos 25\pi -1 \) 60. \( \sin(135°)^2 + \cos(-675°)^2 \)

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

61. \( a = 3.1, b = 5.8, A = 30° \) 62. \( a = 9, b = 40, c = 41 \)

\( \text{sines; } B = 69°, C = 81°, c = 6.1 \) \( \text{cosines; } A = 13°, B = 77°, C = 90° \)

Use synthetic substitution to find \( f(3) \) and \( f(-4) \) for each function. (Lesson 7-4)

63. \( f(x) = 5x^2 + 6x - 17 \) 64. \( f(x) = -3x^2 + 2x - 1 \) 65. \( f(x) = 4x^2 - 10x + 5 \)

46, 39, -22, -57, 11, 109

66. **Physics** A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket \( t \) seconds after firing is given by the formula \( h(t) = -16t^2 + 80t + 200 \). Find the time at which the rocket reaches its maximum height of 300 feet. (Lesson 6-5)

\[ h(t) = -16t^2 + 80t + 200 \]

\[ h(t) = -16(t-5)^2 + 200 \]

**Lesson 13-7 Inverse Trigonometric Functions** 751

51. **Suppose** \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) lie on the line \( y = mx + b \). Then \( m = \frac{y_2 - y_1}{x_2 - x_1} \). The tangent of the angle \( \theta \) the line makes with the positive \( x \)-axis is equal to the ratio \( \frac{y_2 - y_1}{x_2 - x_1} \). Thus \( \tan \theta = m \).
Vocabulary and Concept Check

- This alphabetical list of vocabulary terms in Chapter 13 includes a page reference where each term was introduced.

- Assessment A vocabulary test/review for Chapter 13 is available on p. 830 of the Chapter 13 Resource Masters.

Lesson-by-Lesson Review

For each lesson,
- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

Vocabulary PuzzleMaker

ELL The Vocabulary PuzzleMaker software improves students' mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

MindJogger Videoquizzes

ELL MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)

13-1 Right Triangle Trigonometry

Concept Summary

- If \( \theta \) is the measure of an acute angle of a right triangle, \( \text{opp} \) is the measure of the leg opposite \( \theta \), \( \text{adj} \) is the measure of the leg adjacent to \( \theta \), and \( \text{hyp} \) is the measure of the hypotenuse, then the following are true.

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan \theta = \frac{\text{opp}}{\text{adj}}, \quad \csc \theta = \frac{\text{hyp}}{\text{opp}}, \quad \sec \theta = \frac{\text{hyp}}{\text{adj}}, \quad \cot \theta = \frac{\text{adj}}{\text{opp}}
\]

Remind students to review the Foldable and make sure that their definitions are accurate and complete. Ask them to check over their notes to make sure they have a diagram for each concept and application that they have worked with in this chapter. Have them make any needed additions.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.
Example

Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

Find $a$. $a^2 + b^2 = c^2$  

$\text{Pythagorean Theorem}$

$\begin{align*}
   a^2 + 11^2 &= 14^2 \\
   a &= \sqrt{14^2 - 11^2} \\
   a &= 8.7 \\

   \text{Solve for } a.
\end{align*}$

$\text{Use a calculator.}$

Find $A$. $\cos A = \frac{11}{14}$  

$\cos A = \frac{\text{adj}}{\text{hyp}}$

$\text{Use a calculator to find the angle whose cosine is } \frac{11}{14}$.

$\text{KEYSTROKES: } \boxed{2nd} \ [\cos^{-1}] \ 11 \ \boxed{+} \ 14 \ \boxed{=} \ \boxed{\text{ENTER}}$  

$38.2132107$

To the nearest degree, $A = 38^\circ$.

Find $B$. $38^\circ + B = 90^\circ$  

$\text{Angles } A \text{ and } B \text{ are complementary.}$

$B = 52^\circ$  

$\text{Solve for } B.$

Therefore, $a = 8.7$, $A = 38^\circ$, and $B = 52^\circ$.

$\text{10–15. See margin.}$

Exercises

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.  

$\text{See Examples 4 and 5 on page 704.}$

10. $c = 16, a = 7$

11. $A = 25^\circ, c = 6$

12. $B = 45^\circ, c = 12$

13. $B = 83^\circ, b = \sqrt{31}$

14. $a = 9, B = 49^\circ$

15. $\cos A = \frac{1}{4}, a = 4$

---

Angles and Angle Measure

Concept Summary

- An angle in standard position has its vertex at the origin and its initial side along the positive $x$-axis.
- The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. If the rotation is in a counterclockwise direction, the measure of the angle is positive. If the rotation is in a clockwise direction, the measure of the angle is negative.

Examples

Rewrite the degree measure in radians and the radian measure in degrees.

1. $240^\circ$

$240^\circ = \frac{240^\circ}{180^\circ} \cdot \frac{\pi \text{ radians}}{180^\circ}$

$= \frac{240\pi}{180}$ radians or $\frac{4\pi}{3}$

2. $\frac{\pi}{12}$

$\frac{\pi}{12} = \frac{\frac{\pi}{12} \text{ radians}}{\frac{180^\circ}{\pi \text{ radians}}}$

$= \frac{\pi}{12} \cdot \frac{180^\circ}{\pi}$

$= \frac{180^\circ}{12}$ or $15^\circ$
Exercise 13-3
Trigonometric Functions of General Angles

Concept Summary
- You can find the exact values of the six trigonometric functions of \( \theta \) given the coordinates of a point \( P(x, y) \) on the terminal side of the angle.

\[
\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}, \quad x \neq 0 \\
\csc \theta = \frac{r}{y}, \quad y \neq 0, \quad \sec \theta = \frac{r}{x}, \quad x \neq 0 \quad \cot \theta = \frac{x}{y}, \quad y \neq 0
\]

Example
Find the exact value of cos 150°.
Because the terminal side of 150° lies in Quadrant II, the reference angle \( \theta' \) is 180° - 150° or 30°. The cosine function is negative in Quadrant II, so cos 150° = -cos 30° or \(-\sqrt{3}/2\).

Exercise 13-4
Law of Sines

Concept Summary
- You can find the area of \( \triangle ABC \) if the measures of two sides and their included angle are known.

\[
\text{area} = \frac{1}{2}bc \sin A \quad \text{area} = \frac{1}{2}ac \sin B \quad \text{area} = \frac{1}{2}ab \sin C
\]
- Law of Sines:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]
Example 3. Solve \( \triangle ABC \).

First, find the measure of the third angle.

\[
53^\circ + 72^\circ + B = 180^\circ \quad \text{The sum of the angle measures is } 180^\circ.
\]

\[
B = 55^\circ \quad 180 - (53 + 72) = 55
\]

Now use the Law of Sines to find \( b \) and \( c \). Write two equations, each with one variable.

\[
\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{Law of Sines}
\]

\[
\frac{\sin 53^\circ}{20} = \frac{\sin 72^\circ}{c}
\]

\[
c = \frac{20 \sin 72^\circ}{\sin 53^\circ}
\]

\[
c = 23.8
\]

Therefore, \( B = 55^\circ, b = 20.5 \), and \( c = 23.8 \).

31, 32, 34, 35. See margin.

Exercises

Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. See Examples 3–5 on pages 727 and 728.

30. \( a = 24, b = 36, A = 64^\circ \) \( \text{no} \)

31. \( A = 40^\circ, b = 10, a = 8 \)

32. \( b = 10, c = 15, C = 66^\circ \)

33. \( A = 82^\circ, a = 9, b = 12 \) \( \text{no} \)

34. \( A = 105^\circ, a = 18, b = 14 \)

35. \( B = 46^\circ, C = 83^\circ, b = 65 \)

13-5

Law of Cosines

Concept Summary

- Law of Cosines: 
  \[
  a^2 = b^2 + c^2 - 2bc \cos A
  \]

\[
b^2 = a^2 + c^2 - 2ac \cos B
\]

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

Example 4. Solve \( \triangle ABC \) for \( A = 62^\circ, b = 15, \) and \( c = 12 \).

You are given the measure of two sides and the included angle. Begin by drawing a diagram and using the Law of Cosines to determine \( a \).

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

\[
a^2 = 15^2 + 12^2 - 2(15)(12) \cos 62^\circ
\]

\[
a^2 = 200
\]

\[
a = 14.1
\]

Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle \( C \).

\[
\sin 62^\circ = \frac{\sin C}{14.1}
\]

\[
\sin C = \frac{12 \sin 62^\circ}{14.1} \quad \text{or about } 48.7^\circ
\]

Use a calculator.

The measure of the angle \( B \) is approximately \( 180 - (62 + 48.7) \) or \( 69.3^\circ \).

Therefore, \( a = 14.1, C = 48.7^\circ, B = 69.3^\circ \).
Answers
36. cosines; 4.6, \( A = 84^\circ, B = 61^\circ \)
37. sines; \( C = 105^\circ, a = 28.3, c = 38.6 \)
38. cosines; \( A = 45^\circ, B = 58^\circ, C = 77^\circ \)
39. cosines; \( A = 33^\circ, B = 82^\circ, c = 6.4 \)
40. sines; \( B = 52^\circ, C = 92^\circ, a = 10.2; B = 128^\circ, C = 16^\circ, c = 2.8 \)
41. cosines; \( B = 26^\circ, C = 125^\circ, a = 8.3 \)

Exercises
Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. **See Examples 1 and 2 on pages 734 and 735.**

36. \( \cos \) 37. \( \cos \) 38. \( \sin \) 39. \( \cos \) 40. \( \sin \) 41. \( \cos \)

36. \( \cos \) 37. \( \sin \) 38. \( \cos \)

39. \( C = 65^\circ, a = 4, b = 7 \)
40. \( A = 36^\circ, a = 6, b = 8 \)
41. \( b = 7.6, c = 14.1, A = 29^\circ \)

**13-6 Circular Functions**

**Concept Summary**
- If the terminal side of an angle \( \theta \) in standard position intersects the unit circle at \( P(x, y) \), then \( \cos \theta = x \) and \( \sin \theta = y \). Therefore, the coordinates of \( P \) can be written as \( P(\cos \theta, \sin \theta) \).

**Example**
Find the exact value of \( \cos \left( -\frac{7\pi}{4} \right) \).

\[
\cos \left( -\frac{7\pi}{4} \right) = \cos \left( -\frac{7\pi}{4} + 2\pi \right) = \cos \frac{\pi}{4} \text{ or } \frac{\sqrt{2}}{2}
\]

**Exercises**
Find the exact value of each function. **See Example 2 on page 741.**

42. \( \sin (-150^\circ) = -\frac{1}{2} \)
43. \( \cos 300^\circ = \frac{1}{2} \)
44. \( \sin 45^\circ \cdot \sin 225^\circ = -\frac{1}{2} \)
45. \( \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \)
46. \( (\sin 30^\circ)^2 + (\cos 30^\circ)^2 = 1 \)
47. \( 4 \cos 150^\circ + 2 \sin 300^\circ = -\sqrt{3} \)

**13-7 Inverse Trigonometric Functions**

**Concept Summary**
- \( y = \sin x \) if and only if \( y = \sin x \) and \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \).
- \( y = \cos x \) if and only if \( y = \cos x \) and \( 0 \leq x \leq \pi \).
- \( y = \tan x \) if and only if \( y = \tan x \) and \( -\frac{\pi}{2} < x < \frac{\pi}{2} \).

**Example**
Find the value of \( \cos^{-1} \left[ \tan \left( -\frac{\pi}{6} \right) \right] \) in radians. Round to the nearest hundredth.

**KEYSTROKES:**

\[ \boxed{2nd \ [\cos^{-1}] \ \tan \ (-) \ \boxed{2nd \ [\sin]} \ \boxed{6} \ \boxed{\pi} \ \boxed{6} \ \boxed{\pi} \ \boxed{\cos^{-1}} \ \boxed{-} \ \boxed{\tan} \ \boxed{-} \ \boxed{\pi} \ \boxed{6} \ \boxed{\cos^{-1}} \ \boxed{2} \ \boxed{1} \ \boxed{8} \ \boxed{6} \ \boxed{276035} \]

Therefore, \( \cos^{-1} \left[ \tan \left( -\frac{\pi}{6} \right) \right] = 2.19 \) radians.

**Exercises**
Find each value. Write angle measures in radians. Round to the nearest hundredth. **See Example 3 on page 748.**

48. \( \sin^{-1} (-1) = -\pi \)
49. \( \tan^{-1} \sqrt{3} = 1.05 \)
50. \( \tan^{-1} \left( \frac{\arcsin \frac{3}{5}}{2} \right) = 0 \)
51. \( \cos (\sin^{-1} 1) = 0 \)
1. **Draw** a right triangle and label one of the acute angles \( \theta \). Then label the hypotenuse \( \text{hyp} \), the side opposite \( \text{opp} \), and the side adjacent \( \text{adj} \). See margin.

2. **State** the Law of Sines for \( \triangle ABC \). \[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

3. **Describe** a situation in which you would solve a triangle by first applying the Law of Cosines. Sample answer: when the measures of two sides and the included angle are given.

**Vocabulary and Concepts**

**Skills and Applications**

Solve \( \triangle ABC \) by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

4. \( a = 7, \ A = 49^\circ, \ b = 6.1, \ c = 9.3, \ B = 41^\circ \)

5. \( B = 75^\circ, \ b = 6 \)

6. \( A = 22^\circ, \ c = 8, \ a = 3.0, \ b = 7.4, \ B = 68^\circ \)

7. \( a = 7, \ c = 16 \)

8. \( a = 1.6, \ c = 6.2, \ A = 15^\circ \)

9. \( b = 14.4, \ A = 26^\circ, \ B = 64^\circ \)

Rewrite each degree measure in radians and each radian measure in degrees.

10. \( \frac{11\pi}{2}, \ 990^\circ \)

11. \( 330^\circ, \ \frac{11\pi}{6} \)

12. \( -600^\circ, \ -\frac{10\pi}{3} \)

13. \( -\frac{7\pi}{4}, \ -315^\circ \)

Find the exact value of each expression. Write angle measures in degrees.

14. \( \cos (-120^\circ) \)

15. \( \sin \frac{7\pi}{4} \)

16. \( \cot 300^\circ \)

17. \( \sec \left( -\frac{7\pi}{6} \right) \)

18. \( \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) \)

19. \( \arctan 1 \)

20. \( \tan 135^\circ \)

21. \( \csc \frac{5\pi}{6} \)

22. Determine the number of possible solutions for a triangle in which \( A = 40^\circ, \ b = 10, \) and \( a = 14 \). If a solution exists, solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. **one: \( B = 27^\circ, \ C = 113^\circ, \ c = 20.0 \)**

23. Suppose \( \theta \) is an angle in standard position whose terminal side lies in Quadrant II. Find the exact values of the remaining five trigonometric functions for \( \theta \) for \( \cos \theta = -\frac{\sqrt{3}}{2} \).

24. GEOLGY From the top of the cliff, a geologist spots a dry riverbed. The measurement of the angle of depression to the riverbed is 70°. The cliff is 50 meters high. How far is the riverbed from the base of the cliff? **18.2 m**

25. STANDARDIZED TEST PRACTICE Triangle \( \triangle ABC \) has a right angle at \( C \), angle \( B = 30^\circ \), and \( BC = 6 \). Find the area of triangle \( \triangle ABC \).

**Portfolio Suggestion**

**Introduction** In this chapter, you studied a number of different approaches to naming and measuring angles and their trigonometric functions.

**Ask Students** Of the seven lessons in this chapter, pick the one that you are still having some trouble understanding. Describe what questions you still have about this lesson. Explain how this lesson might have been explained in a way that would be clearer to you. Place this in your portfolio.
Part 1  Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. If \(3n + k = 30\) and \(n\) is a positive even integer, then which of the following statements must be true? C
   I. \(k\) is divisible by 3.
   II. \(k\) is an even integer.
   III. \(k\) is less than 20.
   A. I only        B. II only
   C. I and II only   D. I, II, and III

2. If \(4x^2 + 5x = 80\) and \(4x^2 - 5y = 30\), then what is the value of \(6x + 6y\)? C
   \(\begin{align*}
   A &: 10 \\
   B &: 50 \\
   C &: 60 \\
   D &: 110
   \end{align*}\)

3. If \(a = b + cb\), then what does \(\frac{b}{a}\) equal in terms of \(c\)? B
   \(\begin{align*}
   A &: \frac{1}{c} \\
   B &: \frac{1}{1+c} \\
   C &: 1 - c \\
   D &: 1 + c
   \end{align*}\)

4. What is the value of \(\sum_{n=1}^{5} 3n^2\)? D
   A. 55  \\
   B. 58  \\
   C. 75  \\
   D. 165

5. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble? D
   A. 4        B. 6        C. 8        D. 12

6. From a lookout point on a cliff above a lake, the angle of depression to a boat on the water is \(12^\circ\). The boat is 3 kilometers from the shore just below the cliff. What is the height of the cliff from the surface of the water to the lookout point? D

7. If \(x + y = 90^\circ\) and \(x\) and \(y\) are positive, then
   \(\text{cos } x = \frac{1}{2}\).
   A. 0  \\
   B. \(\frac{1}{2}\)  \\
   C. 1  \\
   D. cannot be determined

8. A child flying a kite holds the string 4 feet above the ground. The taut string is 40 feet long and makes an angle of \(35^\circ\) with the horizontal. How high is the kite off the ground? A
   \(\begin{align*}
   A &: 4 + 40 \sin 35^\circ \\
   B &: 4 + 40 \cos 35^\circ \\
   C &: 4 + 40 \tan 35^\circ \\
   D &: 4 + \frac{40}{\sin 35^\circ}
   \end{align*}\)

9. If \(\sin \theta = -\frac{1}{2}\) and \(180^\circ < \theta < 270^\circ\), then \(\theta = B\)
   \(\begin{align*}
   A &: 200^\circ. \\
   B &: 210^\circ. \\
   C &: 225^\circ. \\
   D &: 240^\circ.
   \end{align*}\)

10. If \(\cos \theta = -\frac{8}{17}\) and the terminal side of the angle is in quadrant IV, then \(\theta = C\)
   \(\begin{align*}
   A &: -\frac{15}{8}. \\
   B &: -\frac{17}{15}. \\
   C &: -\frac{15}{17}. \\
   D &: -\frac{15}{17}.
   \end{align*}\)
Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. The length, width, and height of the rectangular box illustrated below are each integers greater than 1. If the area of \(ABCD\) is 18 square units and the area of \(CDEF\) is 21 square units, what is the volume of the box? 126 units\(^3\)

![Rectangular Box Diagram]

12. When six consecutive integers are multiplied, their product is 0. What is their greatest possible sum? 15

13. The average (arithmetic mean) score for the 25 players on a team is \(n\). Their scores range from 60 to 100, inclusive. The average score of 20 of the players is 70. What is the difference between the greatest and least possible values of \(n\)? 8

14. The variables \(a\), \(b\), \(c\), \(d\), and \(e\) are integers in a sequence, where \(a = 2\) and \(b = 12\). To find the next term, double the last term and add that result to one less than the next-to-last term. For example, \(c = 25\), because \(2(12) = 24\), \(2 + 1 = 1\), and \(24 + 1 = 25\). What is the value of \(e\)? 146

15. In the figure, if \(t = 2v\), what is the value of \(x\)? 150

![Triangle Diagram]

16. If \(b = 4\), then what is the value of \(a\) in the equations below? 5
   \[3a + 4b + 2c = 33\]
   \[2b + 4c = 12\]

17. At the head table at a banquet, 3 men and 3 women sit in a row. In how many ways can the row be arranged so that the men and women alternate? 72

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

A. the quantity in Column A is greater,
B. the quantity in Column B is greater,
C. the two quantities are equal, or
D. the relationship cannot be determined from the information given.

![Quantitative Comparison Table]

18. A container holds a certain number of tiles. The tiles are either red or white. One tile is chosen from the container at random.

   - probability of choosing a red or a white tile | 200%
   - A

19. \(x = (4x)^4 + \frac{x}{4}\)

   - 5
   - \(\frac{1}{4}\)
   - A

20. The area of square \(ABCD\) is 64 units\(^2\).
   - area of circle \(O\) | 192 units\(^2\)
   - B

21. \(PQRS\) is a square.
   - \(OS\)
   - 2
   - B

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9. \( f(x) = x + 3 \)

10. \( f(x) = 5x + 2 \)

11. \( f(x) = x^2 - 4 \)

12. \( f(x) = -7x - 9 \)

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15. \( \sin \theta = \frac{4}{11}; \cos \theta = \frac{\sqrt{105}}{11}; \tan \theta = \frac{4\sqrt{105}}{105}; \csc \theta = \frac{11}{4}; \sec \theta = \frac{11\sqrt{105}}{105}; \cot \theta = \frac{\sqrt{105}}{4} \)

16. \( \sin \theta = \frac{3}{5}; \cos \theta = \frac{4}{5}; \tan \theta = \frac{3}{4}; \csc \theta = \frac{5}{4}; \sec \theta = \frac{5}{4}; \cot \theta = \frac{4}{3} \)

17. \( \sin \theta = \frac{\sqrt{7}}{4}; \cos \theta = \frac{3}{4}; \tan \theta = \frac{\sqrt{7}}{3}; \csc \theta = \frac{4\sqrt{7}}{7}; \sec \theta = \frac{4}{3}; \cot \theta = \frac{3\sqrt{7}}{7} \)

18. \( \sin \theta = \frac{9\sqrt{106}}{106}; \cos \theta = \frac{5\sqrt{106}}{106}; \tan \theta = \frac{9}{5}; \csc \theta = \frac{\sqrt{106}}{9}; \sec \theta = \frac{\sqrt{106}}{5}; \cot \theta = \frac{5}{9} \)

19. \( \sin \theta = \frac{\sqrt{5}}{5}; \cos \theta = \frac{2\sqrt{5}}{5}; \tan \theta = \frac{1}{2}; \csc \theta = \sqrt{5}; \sec \theta = \frac{\sqrt{5}}{2}; \cot \theta = 2 \)

20. \( \sin \theta = \frac{\sqrt{15}}{8}; \cos \theta = \frac{7}{8}; \tan \theta = \frac{\sqrt{15}}{7}; \csc \theta = \frac{8\sqrt{15}}{15}; \sec \theta = \frac{8}{7}; \cot \theta = \frac{7\sqrt{15}}{15} \)

27a. \( \sin 30^\circ = \frac{\text{opp}}{\text{hyp}} \quad \text{sine ratio} \)

\( \sin 30^\circ = \frac{x}{2x} \quad \text{Replace opp with x and hyp with 2x.} \)

\( \sin 30^\circ = \frac{1}{2} \quad \text{Simplify.} \)

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19. \( \tan 45^\circ = \frac{\text{adj}}{\text{hyp}} \quad \text{tangent ratio} \)

\( \tan 45^\circ = \frac{x}{x} \quad \text{Replace opp with x and adj with x.} \)

\( \tan 45^\circ = 1 \quad \text{Simplify.} \)
61. Student answers should include the following.

- An angle with a measure of more than 180° gives an indication of motion in a circular path that ended at a point more than halfway around the circle from where it started.
- Negative angles convey the same meaning as positive angles, but in an opposite direction. The standard convention is that negative angles represent rotations in a clockwise direction.
- Rates over 360° per minute indicate that an object is rotating or revolving more than one revolution per minute.

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Algebra Activity
7. To find the length of the apothem, you need to write this equation: \( \cos \theta = \frac{a}{\text{length of radius}} \). If the radius is 1, then \( \cos \theta = a \). If the radius is not 1, then \( a = \text{length of radius} \cdot \cos \theta \).

Pages 722-723, Lesson 13-3
7. 
8. 
17. \( \sin \theta = \frac{24}{25}, \cos \theta = \frac{7}{25}, \tan \theta = \frac{24}{7}, \csc \theta = \frac{25}{24}, \sec \theta = \frac{25}{7}, \cot \theta = \frac{7}{24} \)
18. \( \sin \theta = \frac{\sqrt{5}}{5}, \cos \theta = \frac{2\sqrt{5}}{5}, \tan \theta = \frac{1}{2}, \csc \theta = \sqrt{5}, \sec \theta = \frac{\sqrt{5}}{2}, \cot \theta = 2 \)
19. \( \sin \theta = \frac{8\sqrt{89}}{89}, \cos \theta = \frac{5\sqrt{89}}{89}, \tan \theta = \frac{8}{5}, \csc \theta = \frac{\sqrt{89}}{89}, \sec \theta = \frac{89}{5}, \cot \theta = \frac{5}{8} \)
20. \( \sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5}, \tan \theta = -\frac{3}{4}, \csc \theta = -\frac{5}{3}, \sec \theta = \frac{4}{3}, \cot \theta = \frac{3}{4} \)
21. \( \sin \theta = -1, \cos \theta = 0, \tan \theta = \text{undefined}, \csc \theta = -1, \sec \theta = \text{undefined}, \cot \theta = 0 \)
22. \( \sin \theta = 0, \cos \theta = -1, \tan \theta = 0, \csc \theta = \text{undefined}, \sec \theta = -1, \cot \theta = \text{undefined} \)
23. \( \sin \theta = -\frac{\sqrt{2}}{2}, \cos \theta = \frac{\sqrt{2}}{2}, \tan \theta = -1, \csc \theta = -\sqrt{2}, \sec \theta = \sqrt{2}, \cot \theta = -1 \)
24. \( \sin \theta = -\frac{\sqrt{6}}{3}, \cos \theta = -\frac{\sqrt{3}}{3}, \tan \theta = \sqrt{2}, \csc \theta = -\frac{\sqrt{6}}{2}, \sec \theta = -\sqrt{3}, \cot \theta = \frac{\sqrt{2}}{2} \)
25. 
26. 
27. 
28. 

Chapter 13 Additional Answers
Pages 736–737, Lesson 13-5

35. Answers should include the following.
   - The Law of Cosines can be used when you know all three sides of a triangle or when you know two sides and the included angle. It can even be used with two sides and the nonincluded angle. This set of conditions leaves a quadratic equation to be solved. It may have one, two, or no solution just like the SSA case with the Law of Sines.
   - Given the latitude of a point on the surface of Earth, you can use the radius of the Earth and the orbiting height of a satellite in geosynchronous orbit to create a triangle. This triangle will have two known sides and the measure of the included angle. Find the third side using the Law of Cosines and then use the Law of Sines to determine the angles of the triangle. Subtract 90 degrees from the angle with its vertex on Earth’s surface to find the angle at which to aim the receiver dish.

Page 744, Lesson 13-6

34.